The complexity of computable entailment

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Foundational analysis

Suppose we're committed to a particular foundational programme of limited strength, such as predicativism or finitistic reductionism.

1. How do we know which theorems we're entitled to assert?
2. How do we know what mathematics we're giving up?

Let's call the process of answering these questions *foundational analysis*.

Reverse mathematics can help

If we formalise our foundation in second order arithmetic, results in reverse mathematics will let us know which theorems we're entitled to assert and which remain out of reach.

This is done by proving *equivalences* between such theorems and subsystems of second order arithmetic, over a weak base theory.

Reverse mathematics looks like it's just what we need to carry out foundational analysis.

Syntax and semantics of second order arithmetic

Second order arithmetic, L_2, is a two-sorted first order system with number variables $m, n, i, \ldots$ and set variables $X, Y, Z, \ldots$ ranging over subsets of the domain.

L_2-structures are models of the first order language of arithmetic extended with a collection of sets for the second order variables to range over:

$$\mathcal{M} = \langle M, S, +, \cdot, <, 0, 1 \rangle$$

where $S \subseteq \mathcal{P}(M)$.

Axioms of second order arithmetic

The axioms of second order arithmetic or Z_2 are the universal closures of the following:

- PA^-, the axioms for a discrete ordered semiring.
- Induction axiom:

$$(0 \in X \land \forall n(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n(n \in X).$$

- Comprehension scheme: the universal closures of all formulae of the form

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n))$$

with X not free in φ.

Subsystems of second order arithmetic

Subsystems of Z_2 are obtained by

1. Restricting the comprehension scheme to particular syntactically defined subclasses
2. Adding other closure conditions like transfinite recursion.

Subsystems of Z_2	**Defining conditions**
RCA_0	Recursive (Δ^0_1) comprehension
WKL_0	RCA_0 plus weak König's lemma
ACA_0	Arithmetical comprehension
ATR_0	ACA_0 plus arithmetical transfinite recursion
$\Pi^1_1{-}CA_0$	Π^1_1 comprehension

Foundational programmes and the Big Five

The most important subsystems of second order arithmetic, known as the Big Five, formally capture some philosophically-motivated programmes in foundations of mathematics.

Foundational programmes	**Subsystems of Z_2**
Constructivism (Bishop)[1]	RCA_0
Finitistic reductionism (Hilbert)	WKL_0
Predicativism (Weyl, Feferman)	ACA_0
Predicative reductionism (Friedman, Simpson)	ATR_0
Impredicativity (Feferman *et al.*)	$\Pi^1_1{-}CA_0$

[1] This identification is problematic since reverse mathematics is classical.

Computational reverse mathematics

Developed by Richard Shore in two recent papers (Shore 2010, 2011), computational reverse mathematics draws on recursion theory rather than proof theory.

It has a two-fold motivation:

- Giving an account of reverse mathematics which most mathematicians will find natural, in computational and construction-oriented terms.
- Extending reverse mathematical analysis from countable structures to uncountable ones.

The main question

Can computational reverse mathematics be used to carry out the foundational analysis outlined at the beginning?

To answer this, we first need to look at the details of Shore's programme.

ω-models

Computational reverse mathematics builds on a tradition of looking at ω-*models*, structures which extend the standard model of arithmetic $\mathbb{N}$.

- First order part is the natural numbers $\omega = \{0, 1, 2, \dots\}$.
- Second order part $\mathcal{C} \subseteq \mathcal{P}(\omega)$ closed under particular recursion-theoretic operations.
- Closure under more operations $\Leftrightarrow$ model of stronger theories.

Turing ideals

These models are also known as *Turing ideals*. All Turing ideals are models of RCA_0. Turing ideals satisfying stronger closure conditions are also models of stronger theories such as ACA_0.

Definition (Turing ideal)

Let $\mathcal{C}$ be a nonempty subset of $\mathcal{P}(\omega)$ closed under Turing reducibility and recursive joins. Then we call $\mathcal{C}$ a *Turing ideal*.

A set X is *Turing reducible* to a set Y, $X \leq_{\mathbf{T}} Y$, iff there is a Turing machine with an oracle for Y which computes X.

The *recursive join* of two sets X and Y is given by

$$X \oplus Y = \{2n : n \in X\} \cup \{2n+1 : n \in Y\}.$$

Closure conditions and subsystems of Z_2

There is a close connection between recursion-theoretic principles and the major subsystems of second order arithmetic.

Recursion–theoretic principles	**Subsystems of Z_2**
Turing reducibility and recursive joins	RCA_0
Jockush–Soare low basis theorem	WKL_0
Turing jump	ACA_0
Hyperarithmetic reducibility	ATR_0
Hyperjump	$\Pi^1_1-CA_0$

Computable entailment and equivalence

Traditional reverse mathematics looks for *provable equivalences* over a base theory. Computational reverse mathematics looks for *computable equivalences*.

Definition (Computable entailment and equivalence)

Let $\mathcal{C}$ be a Turing ideal, and let φ be a sentence of second order arithmetic. $\mathcal{C}$ *computably satisfies* φ if φ is true in the ω-model whose second order part consists of $\mathcal{C}$.

A sentence ψ *computably entails* φ, $\psi \models_{\mathcal{C}} \varphi$, if every Turing ideal $\mathcal{C}$ satisfying ψ also satisfies φ.

Two sentences ψ and φ are *computably equivalent*, $\psi \equiv_{\mathcal{C}} \varphi$, if each computably entails the other.

These definitions extend to theories in the obvious way.

The complexity of computable entailment

The provability relation $\vdash$ is epistemically tractable, since in order to show that ψ follows from φ, we merely need to exhibit a finite proof witness.

Semantic relations like truth are far more complicated, and as we might expect, so is computable entailment.

Theorem (Mummert)

Computable entailment is Π^1_1 *complete.*

Predicativism's incompatibility with CRM

Predicativism in the philosophy of mathematics is a well-established position. An early proponent was Hermann Weyl, and the view has been much developed by Solomon Feferman.

The predicativist holds that, given some domain X, only those sets exist which can be defined without employing quantifiers ranging over collections of which X is a member, like $\mathcal{P}(X)$.

In the case of the natural numbers, only those sets $Y \subseteq \mathbb{N}$ exist which are defined by formulae that do not employ any second order quantifiers: in other words, arithmetical comprehension or ACA_0.

A Π^1_1 complete relation such as computable entailment is impredicative. So a committed predicativist will not be persuaded by arguments that invoke foundational analysis performed using computational reverse mathematics, since its theoretical commitments outstrip those countenanced by the predicativist.

Extending the argument to ATR_0

Fact

The existence of a Π^1_1 complete set is equivalent over ACA_0 to $\Pi^1_1{-}\mathrm{CA}_0$ (without set parameters).

So the commitment to the computable entailment relation outstrips not just predicativism (ACA_0) but also the stronger system ATR_0.

ATR_0 is Π^1_1 conservative over Feferman's system IR of predicative analysis, so we can think of ATR_0 as a formalisation in second order arithmetic of *predicative reductionism*, a view developed by Friedman and Simpson.

So the theoretical commitments which accompany the use of computable entailment outstrip those acceptable to partisans of most of the foundational programmes analysable in reverse mathematics.

Uncountable reverse mathematics

Shore (2011) develops one way to pursue a reverse mathematics of uncountable structures. This extends computational reverse mathematics by replacing the underlying notion of Turing computability with admissible recursion (α-recursion) on initial segments of Gödel's L.

We use the second order language of set theory and assume that $V = L$. Fixing an uncountable cardinal κ, let L_κ be an initial segment of the constructible universe. Call a nonempty set $\mathcal{S} \subseteq \mathcal{P}(L_\kappa)$ *κ-closed* iff it is closed under κ-reducibility and effective joins.

Definition

Let κ be an uncountable cardinal and let φ, ψ be sentences of $\mathrm{L}^2_\in$.
φ *κ-computably entails* ψ iff for every κ-closed $\mathcal{S} \subseteq \mathcal{P}(L_\kappa)$,
if $(L_\kappa, \mathcal{S}) \models \varphi$ then $(L_\kappa, \mathcal{S}) \models \psi$.

Complexity revisited

Now we have a notion of computable entailment for κ-recursion, we can analyse the complexity of the relation and obtain an analogous result.

Theorem

Computable entailment for uncountable κ-recursion is Π^1_1 complete.

The proof of this result, modulo some coding details, is essentially the same as the classical one.

Thank you.

References

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