# Formal Methods Seminar 2012: The compactness and Löwenheim–Skolem theorems

Benedict Eastaugh*

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First-order logic is a powerful and expressive system which has been used to formalise many basic systems of mathematics, including set theory, arithmetic, and the real closed field. It has been intensively studied since the early 20th century. Here we present two of the most important metalogical results about first-order logic: the compactness and Löwenheim–Skolem theorems.

As well as defining the notions required to state and prove these theorems, we shall examine some of their philosophical impact, which has made itself felt not only in logic and philosophy of mathematics but also in the philosophy of language.

## 1 The compactness theorem

The compactness theorem was originally proved by Gödel [1930] as a corollary to his completeness theorem for first-order logic.

**Lemma 1.1** (Model Existence Lemma). *Let $L$ be a first-order language. If $S$ is a consistent theory then $S$ has a model of cardinality $\kappa$, where $\kappa \leq |L|$.*

**Theorem 1.2** (Compactness Theorem). *Let $S$ be a set of wffs of first-order logic. If every finite subset of $S$ is satisfiable, then $S$ is satisfiable.*

*Proof.* Consider the contraposition of compactness: if $S$ is not satisfiable, then there is a finite subset of $S$ which is not satisfiable. So assume that $S$ is not satisfiable: there is no structure $A$ such that $A \models S$. We thus merely need to prove that there exists a finite subset $S' \subseteq S$ such that $S'$ is not satisfiable.

By lemma 1.1, if $S$ is consistent then $S$ has a model. So if $S$ has no model then $S$ is inconsistent. For a theory to be inconsistent is just for it to prove a contradiction—that is, there is some wff $\psi$ such that $S \vdash \psi \wedge \neg\psi$. Consider a proof witnessing this statement,

$$\Theta = \langle \theta_1, \ldots, \theta_n \rangle \tag{1}$$

for $n \in \mathbb{N}$. Since $\Theta$ must be finitely long, only finitely many $\theta_i$ can appear in it such that $\theta_i \in S$. Let $S'$ be the set of wffs of $S$ which appear in $\Theta$. $S'$ must also be finite, and of course $S' \vdash \psi \wedge \neg\psi$, so $S'$ is inconsistent, and thus is a finite subset of $S$ which is not satisfiable. □

There are also purely model-theoretic proofs of the compactness theorem—see for example Hodges [1997, pp. 125–127].

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### 1.1 Applications of compactness

The compactness theorem is a model-existence theorem: it says that *given* that some condition holds, *there exists a model* with certain properties. A simple application of the theorem is showing that given the consistency of some theory, non-standard models of that theory exists. This is easily seen in the case of arithmetic; the theory PA is just the ordinary first-order theory of Peano Arithmetic.

**Lemma 1.3.** *There exists a model of PA which is not isomorphic to the standard model.*

*Proof.* Add a new constant symbol $e$ to the language of arithmetic and for each numeral $\bar{n}$ add the sentence $\bar{n} < e$ to the axioms of PA to obtain the new theory $S$. Any finite $T \subseteq S$ is satisfied by the usual natural number structure $\mathbb{N}$ by interpreting the constant $e$ as some number larger than any numeral employed in the sentences of $T$. So by compactness, $S$ has a model $M$ which contains an element $e^M$ greater than any standard natural number. As PA is a subtheory of $S$, $M$ is also a model of PA. □

Compactness is also useful for proving that certain classes of structures are finitely axiomatisable; for details see van Dalen [2004, pp. 114–116]. However, the best-known application of compactness is in proving the upward direction of the *Löwenheim–Skolem theorem.*

## 2 The Löwenheim–Skolem theorems

From the compactness theorem we first learn about non-standard models: structures which satisfy a first-order theory $S$ and yet are not isomorphic to the intended interpretation. Non-standard models of arithmetic bring this out quite forcefully: these strange structures are nothing like we naïvely expect from a theory which seems, in its essentials, so transparent.

One aspect of this is cardinality: we think of arithmetic as inherently describing a countable structure, and indeed our very notion of a set being countable is that it can be put into a one-to-one correspondence with the natural numbers. But not only are there uncountable models of first-order theories such as Peano Arithmetic, but there are arbitrarily large models of every first-order theory.

Perhaps even more surprisingly, no matter how large the intended model of a first-order theory is, it also has a countable model. This result was originally proved in 1915 by the German mathematician Leopold Löwenheim. A simpler and more general proof was given in 1920 by the Norwegian logician Thoralf Skolem [Löwenheim 1915, Skolem 1920].

### 2.1 Some model theory

In order to state the theorem in its full generality we need a few definitions from model theory, which allow us to characterise languages and structures in a more fine-grained way. By convention for a structure $A$ we add a superscript to denote an element, function or relation belonging to that structure. For instance, given a constant $c$ we would denote its referant in $A$ by $c^A$.

#### 2.1.1 Languages, names and theories

The languages we have discussed so far have all been *countable*: their formulae can be enumerated by assigning a unique natural number to each formula. However, it is often convenient to be able to add new non-logical symbols to a language, perhaps even uncountably many of them. Under some circumstances we thus need to pay heed to the cardinality of a language: how many different symbols does it have, and how many formulae does it include?

**Definition 2.1.** Two formulae are *variants* of one another if they differ only in their choice of variable names—that is, if one can be obtained from the other by a uniform substitution of variables. $\varphi(x_1, \ldots, x_n)$ is a variant of $\varphi(y_1, \ldots, y_n)$, and vice versa, while $\forall x(P(x))$ is a variant of $\forall y(P(y))$.

**Definition 2.2.** The *cardinality* of a first-order language $L$, $|L|$, is the number of equivalence classes of formulae of $L$ under the relation of being variants.

**Lemma 2.3.** *Let $L$ be a first-order language and let $\kappa$ be the number of non-logical symbols of $L$. Then $|L| = max\{\omega, \kappa\}$.*

**Definition 2.4.** Let $L$ be a first-order language and $A$ an $L$-structure. Then the *theory of $A$*, $Th(A)$, is the set of all sentences $\varphi$ of $L$ such that $A \models \varphi$.

**Definition 2.5.** Suppose we have two signatures, $L^-$ and $L^+$, and that $L^- \subseteq L^+$. Then if $A$ is an $L^+$-structure we can transform it into an $L^-$-structure by forgetting the symbols of $L^+$ which don't appear in $L^-$. This gives us the *$L^-$-reduct* of $A$, $A|L^-$.

**Lemma 2.6.** *Let $A$ and $A|L^-$ be as above. Suppose we have a theory $S$ in the language $L^+$ and a theory $T$ in the language $L^-$ such that $T \subseteq S$. Then if $A|L^- \models T$ then $A \models T$; and if $A \models S$ then $A|L^- \models T$.*

The proofs of lemmas 2.3 and 2.6 are left as exercises for the reader.

## 2.2 The downwards Löwenheim–Skolem theorem

**Theorem 2.7.** *Let $L$ be a language of cardinality $\kappa < \lambda$ where $\kappa$ is an infinite cardinal, and let $T$ be a theory in the language of $L$. If $T$ has a model of cardinality $\lambda$ then it also has a model of cardinality $\delta$, with $\kappa \leq \delta < \lambda$.*

*Proof.* Take a set of new constant symbols $C = \{c_\alpha : \alpha < \delta\}$ which don't appear in $L$, and add them to $L$ to obtain the new language $L' = L \cup C$. We then take $T$ and construct a new theory $T'$ by adding new axioms as follows:

$$T' = T \cup \{\text{“}c_\alpha \neq c_\beta\text{”} : \alpha, \beta < \delta \wedge \alpha \neq \beta\}. \tag{2}$$

We now prove that $T'$ is satisfiable. Let $A$ be a model of $T$ with cardinality $\lambda$. From $A$ we obtain a new model $A'$ by adding to it $\delta$ new constants such that $c_\alpha^{A'} \neq c_\beta^{A'}$. We can do this since as $\delta < \lambda$, $\mathrm{dom}(A)$ has a subset of cardinality $\delta$. $A' \models T$ since $A$ is just the $L$-reduct of $A'$, and for all $\alpha, \beta < \delta$ where $\alpha \neq \beta$, $A' \models c_\alpha \neq c_\beta$, so $T'$ is satisfiable.

The cardinality of $L'$ is $\delta$, so by lemma 1.1 $T'$ has a model $B'$ of cardinality $\leq \delta$. But because $B'$ satisfies the additional axioms stating that $c_\alpha \neq c_\beta$ where $\alpha \neq \beta$ and $\alpha, \beta < \delta$, $B'$ has cardinality $\geq \delta$. So the cardinality of $B'$ must be $\delta$.

Finally let $B$ be the $L$-reduct of $B'$, which by lemma 2.6 is a model of $T$ of cardinality $\delta$. $\square$

## 2.3 The upwards Löwenheim–Skolem theorem

Another consequence of compactness is what is often called the *upwards Löwenheim–Skolem theorem*: given any theory $S$ with an infinite model $A$, that theory also has models of every cardinality $\mu > |A|$. The theorem's name is somewhat inappropriate since as we shall see, Skolem rejected the existence of uncountable sets.

**Theorem 2.8.** *Let $L$ be a first-order language with $|L| = \kappa$ where $\kappa$ is an infinite cardinal, and let $T$ be a theory in the language $L$. If $T$ has a model of cardinality $\gamma \geq \kappa$ then for each cardinality $\delta > \gamma$ there exists a model of $T$ with that cardinality.*

We omit the proof of this result since we shall prove a more general version later as theorem 2.15. Combining the upward and downward Löwenheim–Skolem theorems, we can see that if a first-order theory has at least one infinite model, then it has models of every infinite cardinality. This shows that first-order logic is too weak to distinguish between different infinite cardinalities.

If we take the cumulative hierarchy picture at face value then the axioms of ZFC appear to describe a structure containing many infinite ordinals and cardinals. The Löwenheim–Skolem theorem shows that there are many models of ZFC in which the higher reaches of the transfinite do not appear. There are even countable models of ZFC. This seems to fly in the face of the assertion—a theorem of ZFC—that an uncountable set exists. This phenomenon is known as *Skolem's paradox.*

## 2.4 Skolem's paradox

Skolem was a sceptic about the existence of uncountable sets, and he took the Löwenheim–Skolem theorem as evidence that his scepticism was justified and that set-theoretic concepts were inherently relative.

Let's examine one instance of this conceptual relativity, by looking more closely at cardinality. For a set $Y$ to have a greater cardinality than another set $X$ is just for there to be no surjective function $f : X \to Y$. So it is easy to see how $|X| < |Y|$ could be true in a model $M$ of ZFC, just because the model didn't include the necessary surjection, even if there were a larger structure $M'$ which did include it.

Skolem contended that to obtain something absolutely uncountable we would either have to start with uncountably many axioms or with an axiom that could yield uncountably many first-order propositions—in other words, higher-order quantification. However, to take either route would be begging the question, and assuming the existence of uncountable objects in order to prove it.

Jané [2001] is a recent article examining Skolem's view of the relativity of set-theoretic concepts, while Bays [2007] is a thorough interrogation of the mathematics of the paradox.

## 2.5 Putnam's model-theoretic argument

Hilary Putnam [1980] famously put Skolem's paradox to work in the philosophy of language, arguing that the relativity of set-theoretic concepts is also a problem for concepts expressed in natural language, and for scientific theories. Because no first-order theory with an infinite model can determine its interpretation up to isomorphism, the theoretical constraints given by the theory cannot determine reference. Neither can operational constraints since these are "just more theory".

For Putnam, this indeterminacy of interpretation *only* arises given a bad way of thinking about theories. Taking truth to be truth in an intended model and taking intended models to be those satisfying sentences which we want to come out true will always give us too many models. But stipulating sentences to be satisfied is the wrong way to go about fnding intended models, since in order to obtain the sentences that comprise our ideal theory we must already have an interpretation of the language they're expressed in. Thus, even generating Skolemite problems for our own language requires fixity of reference.

These considerations led Putnam to reject metaphysical realism and endorse a Dummetian, anti-realist semantics.

> [T]he world does not pick models or interpret languages. *We* interpret our languages or nothing does.
>
> We need, therefore, a standpoint which links use and reference in just the way that the metaphysical realist standpoint refuses to do. The standpoint of "non-realist semantics" is precisely that standpoint. [Putnam 1980, p. 482]

## 2.6 Generalising Löwenheim–Skolem

The results above can actually be strengthened so that the models obtained are not merely arbitrary models of a given cardinality, but actually bear a close relationship to the models we start with. To do this we look at the relations structures can bear to one another. These notions from model theory are common currency in logic and philosophy of mathematics, and worth learning for their own sake, but here we use them to prove stronger versions of our goal theorems.

### 2.6.1 Extensions and elementarity

We denote $n$-tuples of constants, variables and elements $\langle a_1, \ldots, a_n \rangle$ as $\bar{c}$, $\bar{x}$, $\bar{a}$ and so on. In the case of a function applied to a tuple, $f\bar{a}$, we mean $\langle f(a_1), \ldots, f(a_n) \rangle$.

**Definition 2.9.** Suppose $A$ and $B$ are $L$-structures. An *embedding* $f$ from $A$ into $B$, $f : A \to B$, is a function $f$ from $\mathrm{dom}(A)$ to $\mathrm{dom}(B)$ obeying the following conditions.

1. $f$ is injective.
2. For each constant $c$ of $L$, $f(c^A) = c^B$.
3. For each $n > 0$ and each $n$-ary relation symbol $R$ of $L$ and $n$-tuple $\bar{a}$ of elements of $A$, $\bar{a} \in R^A \leftrightarrow f\bar{a} \in R^B$.
4. For each $n > 0$ and each $n$-ary function symbol $F$ of $L$ and $n$-tuple $\bar{a}$ of elements of $A$, $f(F^A(\bar{a})) = F(f\bar{a})$.

**Definition 2.10.** Given two $L$-structures $A$ and $B$ such that $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$, if the inclusion map $i : \mathrm{dom}(A) \to \mathrm{dom}(B)$ is an embedding then we say that $B$ is an *extension* of $A$, or alternatively that $A$ is a *substructure* of $B$, $A \subseteq B$.

**Definition 2.11.** If an embedding between $L$-structures $f : A \to B$ preserves first-order formulae, $A \models \varphi(\bar{a}) \to B \models \varphi(f\bar{a})$, then we say that $f$ is an *elementary embedding.*

Suppose that $B$ is an extension of $A$. If the inclusion map is an elementary embedding then $B$ is an *elementary extension* of $A$, or equivalently, $A$ is an *elementary substructure* of $B$, $A \preccurlyeq B$.

**Definition 2.12.** Let $A$ be an $L$-structure and $\bar{a}$ be a sequence of all elements of $A$. We choose a sequence $\bar{c}$ of new constants not in $L$ to name the elements of $\bar{a}$. Adjoining these constants to $L$ gives us a new language $L(\bar{c})$ and an $L(\bar{c})$-structure $(A, \bar{a})$ which is just $A$ with the constants $\bar{c}$ interpreted as the constant elements $\bar{a}$.

Then the *elementary diagram of* $A$, $eldiag(A)$, is $Th(A, \bar{a})$. In other words, it's the set of sentences *with parameters from* $A$ that are true in $A$.

**Lemma 2.13.** *Let $L$ be a first-order language and $A$ an $L$-structure. If $B \models eldiag(A)$, then there is an elementary embedding of $A$ into the reduct $B|L$.*

*Proof.* See Hodges 1997, p. 49. □

### 2.6.2 Generalised Löwenheim–Skolem

**Theorem 2.14** (Downwards Löwenheim–Skolem Theorem)**.** *Let $L$ be a first-order language, $A$ an $L$-structure and $\kappa$ an infinite cardinal such that $|L| \leq \kappa \leq |A|$. Then $A$ has an elementary substructure of cardinality $\kappa$.*

*Proof.* Omitted, but see Hodges [1997, pp. 69–72] or van Dalen [2004, p. 124]. $\square$

**Theorem 2.15** (Upwards Löwenheim–Skolem Theorem)**.** *Let $L$ be a first-order language, $A$ an infinite $L$-structure and $\kappa$ a cardinal such that $|L| \leq \kappa$ and $|A| \leq \kappa$. Then $A$ has an elementary extension of cardinality $\kappa$.*

*Proof.* Providing names for the elements of $A$ as necessary, let $T = eldiag(A)$. Add new constants $c_i$ to $L$ for $i < \kappa$, obtaining the language $L^*$. Let $S$ be the theory

$$T \cup \{ c_i \neq c_j : i, j < \kappa \} .$$

Pick an arbitrary finite $U \subseteq S$. Only finitely many of the new constants $c_i$ occur in the sentences of $U$; we denote them $\bar{c}$. Our original structure $A$ is infinite so we simply assign each of the new constants to an element $c_i^A$ of $A$. By this procedure we obtain an $L(\bar{c})$-structure $A'$. Clearly $A' \models U$ so by compactness there is an $L^*$-structure $B$ satisfying $S$.

$B \models eldiag(A)$, so by lemma 2.13 there is an elementary embedding $e : A \to B|L$. Elementary embeddings are injections, so we can straightforwardly replace $e(x^A) = x^B$ in $B|L$ with $x^A$, thus making $B|L$ an elementary extension of $A$.

Since $B \models S$, for every pair of elements $c_i^B$ and $c_j^B$ with $i, j < \kappa$, $c_i^B \neq c_j^B$. The cardinality of $B|L$ is therefore at least $\kappa$. We then apply theorem 2.14 to reduce the cardinality of $B|L$ to exactly $\kappa$. $\square$

## 3 Lindström's theorem

Compactness and the downward Löwenheim–Skolem theorem demonstrate two important properties of first-order logic. The Swedish logician Per Lindström proved in the 1960s that first-order logic is maximal with respect to these properties: it is the strongest logic which has both the compactness property and the downwards Löwenheim–Skolem property. For a proof of the theorem see Väänänen [2010], which is available online.

Put another way, any logic which goes beyond first-order must distinguish between infinite cardinalities: there must be sentences that have models of some infinite cardinalities but not others, and thus make non-trivial assertions about our background ontology, usually taken to be the cumulative hierarchy of sets.

The obvious example of such a logic is second-order logic, which allows us to provide a categorical axiomatisation of the natural number structure, ruling out models of cardinality greater than $\omega$, and also a quasi-categorical axiomatisation of set theory where the only set-sized models are proper initial segments of $V$ satisfying certain closure constraints.

However, second-order logic has significant technical disadvantages compared to first-order logic. In particular, it does not admit a complete proof theory. Many philosophers have also shied away from the additional conceptual and ontological commitments thought to be entailed by quantification over properties, although in recent decades this has changed somewhat and second-order logic has become more acceptable. For a short introduction to second order logic see chapter 4 of van Dalen [2004]. A more expansive technical account and philosophical defence of second order logic is Shapiro [2000].

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