Review of Denis R. Hirschfeldt, *Slicing the Truth*

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Denis R. Hirschfeldt, **Slicing the Truth: On the Computability Theoretic and Reverse Mathematical Analysis of Combinatorial Principles**. Singapore: World Scientific Publishing Company, 2014, *Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore* Vol. 28, pp. xv + 214. ISBN 978-981-4612-61-6 (hardback), $40.

Reverse mathematics is a subfield of mathematical logic devoted to characterising the strength of mathematical theorems from areas such as real and complex analysis, countable algebra, countable infinitary combinatorics, and the topology of complete separable metric spaces. The strength of a theorem φ is determined by proving, over a weak base theory, its equivalence to a known subsystem S of second order arithmetic, thus placing it in a well-studied hierarchy of such systems. The standard base theory for reverse mathematics is RCA_0, whose axioms consist of the axioms of PA^-, plus induction for Σ^0_1 formulas, and the recursive (Δ^0_1) comprehension scheme. A typical equivalence of the sort proved in reverse mathematics is that between the Heine–Borel covering theorem and WKL_0. The Heine–Borel covering theorem (HB) states that every covering of the closed unit interval $[0, 1]$ by a sequence of open intervals has a finite subcovering, while the axioms of WKL_0 consist of those of RCA_0 plus Weak König's Lemma: the statement that every infinite binary tree has an infinite path through it. A standard proof of HB can be formalized within WKL_0. Moreover, by adding HB to the axioms of the base theory RCA_0, one can then "reverse" this result and prove Weak König's Lemma. The Heine–Borel covering theorem is therefore equivalent to WKL_0, provably in RCA_0.

The focus of results in the early years of reverse mathematics was the so-called Big Five subsystems RCA_0, WKL_0, ACA_0, ATR_0, and $\Pi^1_1{-}\mathsf{CA}_0$, and many ordinary mathematical theorems were found to be equivalent to them. The central results are collected in Stephen Simpson's monograph *Subsystems of Second Order Arithmetic* Simpson [2009]. Hirschfeldt's book is a welcome addition to Simpson's classic reference work, providing a fresh and accessible look at a central aspect of contemporary reverse mathematical research.

Tools and concepts from computability theory have been essential to reverse mathematics since its inception, but in recent years they have taken centre stage due to their role in the study of combinatorial principles related to Ramsey's theorem, especially Ramsey's theorem for pairs, RT^2_2. The present volume is an introduction to this ongoing research project. Based on Hirschfeldt's lecture notes, it would serve as an excellent textbook for graduate students who have completed a course on computability theory. In turning his notes into a book Hirschfeldt has retained some of the immediacy of the classroom, and the material is presented in an accessible and lively style. An enjoyable aspect is the large number of questions and exercises, ranging from introductory graduate-level problems to challenging open questions. One minor criticism is that the book has no index. Given the variety of topics covered, and the connections between them, this would have been an extremely useful addition for both students and researchers.

The first, introductory chapter motivates and explains the computability-theoretic and reverse mathematical study of combinatorial problems. Chapter 2 then sets out the basic concepts and techniques of computability theory and forcing. Since the book presumes some background in these areas, chapter 2 is rather brisk, and not suitable as a text for a student learning this material for the first time. The first half of the chapter is concerned with computability theory, and presents the basic notions of computable and computably enumerable functions, relative computability, low and high Turing degrees, and the arithmetical hierarchy. The second half of chapter 2 presents the

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essentials of forcing. While understanding it becomes essential in chapters 6 and 7, this section could be skipped and returned to later.

The main focus of reverse mathematics has been on statements of the form $\forall X[\theta(X) \rightarrow \exists Y \varphi(X, Y)]$ where both θ and φ are formulas of arithmetical complexity, i.e. containing only first order quantifiers ranging over natural numbers. Given a statement P of this form, we say that X is an *instance* of P if $\theta(X)$, and that Y is a *solution* to X if $\varphi(X, Y)$. Many ordinary mathematical statements can be formalized in this way, such as the statement that every continuous function on $[0, 1]$ is bounded, or that every countable commutative ring has a prime ideal. Here, the instance X would be a countable commutative ring, and the solution Y a prime ideal in X.

Given this framing, a natural question concerns the relative complexity of X and Y. In particular, if X is a computable instance of P, what constraints are there on the complexity of Y? Chapter 3 addresses the case where P is Weak König's Lemma, the statement that every infinite subtree of $2^{<\mathbb{N}}$ has an infinite path through it. The answer turns out to be closely related to basis theorems for Π^0_1 classes. A Π^0_1 class consists of the set of paths through a computable infinite binary tree, and a basis theorem for Π^0_1 classes states that, for some class $\mathcal{C}$, every nonempty Π^0_1 class contains a set X such that $X \in \mathcal{C}$. The best known of these basis theorems is that of Jockusch and Soare Jockusch and Soare [1972], which states that every nonempty Π^0_1 class contains a set X which is *low*, i.e. $X' \leq_{\mathrm{T}} \emptyset'$. From the low basis theorem it follows that every computable instance of Weak König's Lemma has a low solution, and moreover that there are models of WKL_0 (the system obtained by adding Weak König's Lemma to RCA_0) which consist only of low sets. This means that WKL_0 is strictly weaker than ACA_0, since every model of ACA_0 contains the halting set, which is not low.

Chapter 4 is devoted to reverse mathematics. Hirschfeldt's presentation is similar to that found in other introductions to reverse mathematics, and the initial part of the chapter applies to the reverse mathematics of any statement which can be formalized in second order arithmetic. In an understandable choice given the book's focus on combinatorial principles which typically only reach the strength of ACA_0, the chapter only examines in detail the systems RCA_0, ACA_0, and WKL_0. Readers interested in understanding the stronger systems also studied in reverse mathematics, such as ATR_0 and $\Pi^1_1{-}\mathsf{CA}_0$, are advised to consult Simpson's textbook Simpson [2009].

While the Big Five are neatly linearly ordered by proof theoretic strength, the world outside them—including many of the principles discussed in this book—appears much more chaotic. Chapter 5, "In Defense of Disarray", is a brief philosophical interlude in which Hirschfeldt professes his preference for this state of affairs, while tempering it with an assertion that "one should never mistake lack of obvious structure for actual lack of structure" (p. 70). He also presses the point, discussed more extensively in the next chapter, that computability theory can give us a finer grained sense of the relationships between mathematical principles than provable equivalence over RCA_0 alone does. The chapter closes with an extended meditation on the relevance of the study of degrees, and in particular of degree classes, for understanding different classes of mathematical objects and the principles that govern them. Although not explained at length, Hirschfeldt's perspective on reverse mathematics and computability theory is an intriguing and sophisticated one. Philosophers of mathematics, as well as logicians, might profit from engaging with it.

With the technical and methdological preliminaries out of the way, chapter 6 introduces the central combinatorial idea of the book, namely Ramsey's theorem, which for Hirschfeldt (p. 75) "captures the idea that total disorder is impossible: a sufficiently large structure will always contain a large ordered substructure". Given $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $[X]^n$ denote the set of n-element subsets of X. For $k \in \mathbb{N}$, a *k-coloring* of $[X]^n$ is a function $c : [X]^n \rightarrow k$. We call $H \subseteq X$ *homogeneous for c* if there is an $i < k$ such that for all $x \in [H]^n$, $c(x) = i$. Ramsey's theorem states that for any k-coloring of $[\mathbb{N}]^n$, where n and k are positive integers, there exists an infinite set which is homogeneous for that coloring.

By fixing $n, k \geq 1$, we can obtain special cases of Ramsey's theorem: RT^n_k states that every k-coloring of $[\mathbb{N}]^n$ has an infinite set homogeneous for it. The following facts are given as an exercise (6.2). For $n \geq 1$ and $k \geq 2$, RCA_0 proves that: RT^1_k; $\mathsf{RT}^{n+1}_k \rightarrow \mathsf{RT}^n_k$; and $\mathsf{RT}^n_k \rightarrow \mathsf{RT}^n_{k+1}$. Hirschfeldt then presents three proofs of Ramsey's theorem and shows, step by step, how to arrive at a uniform proof of RT^n_k that can be carried out in ACA_0 for any $n, k \geq 1$. Jockusch Jockusch [1972] proved the existence of a computable 2-coloring c of $[\mathbb{N}]^3$ such that any infinite set homogeneous for c computes the Turing jump. As a corollary of this result, we also have a reversal: RT^3_2 (and hence RT^n_k for $n \geq 3$ and $k \geq 2$) is equivalent over RCA_0 to ACA_0.

As Hirschfeldt remarks, equivalences of principles P to ACA_0 are often "coarse". The computability-theoretic study of solutions to computable instances of P can reveal a finer structure than reverse mathematics alone, by calculating bounds on their complexity in terms of the arithmetical hierarchy. Jockusch's result that RT^3_2 has a computable instance whose solutions all compute the jump can be relativized to show that for each $n \geq 3$, there is a computable 2-coloring of $[\mathbb{N}]^n$ such that any infinite set homogeneous for c computes $\emptyset^{(n-2)}$. Moreover, for any $n \geq 2$, there exists a computable 2-coloring of $[\mathbb{N}]^n$ with no Σ^0_n-definable infinite homogeneous set. In other words, as n increases, so does the complexity of solutions to computable instances of RT^n_k, as measured in terms of the arithmetical hierarchy.

The only principle left out of the reverse mathematical classification above is RT^2_2, Ramsey's theorem for pairs. Specker Specker [1971] proved that there is a computable 2-coloring of $[\mathbb{N}]^2$ with no computable infinite homogeneous set. This means that RT^2_2 does not hold in the ω-model REC consisting of the computable sets, and since REC is a model of RCA_0, RT^2_2 is not provable in RCA_0. Jockusch Jockusch [1972] improved Specker's result by showing that there is a computable $c : [\mathbb{N}]^2 \to 2$ with no Σ^0_2-definable homogeneous set. Since there are models of WKL_0 consisting entirely of low sets, which are all Σ^0_2, it follows that $\mathsf{WKL}_0 \nvdash \mathsf{RT}^2_2$. Two natural questions are thus: does RT^2_2 imply ACA_0, and if not, does it imply WKL_0?

The first question was answered in the negative by Seetapun's theorem in Seetapun and Slaman [1995], while the second was solved by Liu Liu [2012], who proved that RT^2_2 and WKL_0 are incomparable, in the sense that neither implies the other over RCA_0. A version of Liu's proof appears in the book as an appendix of some ten pages, giving readers an example of a combinatorially complicated forcing argument in reverse mathematics. While it remains far from the easiest example to follow, Hirschfeldt's presentation of the result improves on the original paper, so as well as exposing students to an important and exciting advance in the area, it also provides researchers with an improved reference version of Liu's proof.

An important advance in our understanding of RT^2_2 was made by Cholak, Jockusch, and Slaman Cholak et al. [2001], who split the theorem into a stable part and a cohesive part. We call a coloring $c : [\mathbb{N}]^2 \to k$ *stable* if for all x, $\lim_y c(x, y)$ exists. SRT^2_2 is the statement that every stable 2-coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set. We call a set C *cohesive* for a collection of sets $R_0, R_1, \ldots$ if it is infinite and for all i, either $C - R_i$ or $C \cap R_i$ is finite. COH is the statement that every countable collection of sets has a cohesive set. Neither SRT^2_2 and COH alone imply RT^2_2 over RCA_0 (or even WKL_0), and moreover, neither implies the other. However, the conjunction $\mathsf{SRT}^2_2 + \mathsf{COH}$ is equivalent over RCA_0 to RT^2_2. The colorings used in the proofs of the computability-theoretic results above are often stable, and in fact any 2-coloring can be transformed into a stable coloring using COH. Understanding RT^2_2 in terms of a cohesive part and a stable part provides, amongst other things, new routes to upper and lower bound theorems. It has also opened up new lines of research: principles such as COH are now studied for their own sake, as well as to shed light on existing subjects such as RT^2_2. Both aspects are explored in the book, the former in several sections of chapter 6, and the latter in chapter 9.

Chapter 7 is concerned with conservativity theorems, which play several important roles in the investigation of subsystems of second order arithmetic. Given two theories S and T in languages $\mathcal{L}_S$ and $\mathcal{L}_T$, and a set of sentences Γ drawn from the language $\mathcal{L}_S \cap \mathcal{L}_T$, we say that T is Γ-*conservative* over S if for every $\varphi \in \Gamma$, if $T \vdash \varphi$ then $S \vdash \varphi$. In the special case that Γ consists of all sentences of $\mathcal{L}_S$, we say simply that T is *conservative* over S. Often, a conservativity theorem can help us to determine a system's consistency strength: if a theory T is Π^0_1-conservative over S, then since consistency statements are Π^0_1, S and T are equiconsistent. In reverse mathematics, conservativity theorems have another application: giving uniform proofs of non-implications between particular principles and whole classes of statements. For instance, WKL_0 is Π^1_1 conservative over RCA_0, so no Π^1_1 sentence implies Weak König's Lemma over RCA_0. Moreover, it allows us to understand the relationship between systems of second order arithmetic and systems of first order arithmetic. For example, RCA_0 (and hence also WKL_0) is a conservative extension of $\mathrm{I}\Sigma_1$.

The general strategy for proving that a theory T is Γ-conservative over a theory S is to show that every countable model $\mathcal{M}$ of S can be expanded to a countable model $\mathcal{N}$ of T, while not changing the truth values of any $\varphi \in \Gamma$. These model expansion arguments often use the method of forcing developed in chapter 2. In reverse mathematics, T usually consists of $S + P$, where P

is a sentence of the form $\forall X[\theta(X) \to \exists Y \psi(X,Y)]$ with θ and ψ arithmetical. Starting with an arbitrary countable model $\mathcal{M}$ of S, the standard approach is to use a forcing construction to add a generic G to M such that $M[G] \models S$ and $M[G]$ satisfies an instance of P, while preserving Γ. Iterating the construction and then taking the union produces a model $\mathcal{N} = \mathcal{M}[G_0][G_1]\dots$ of T such that for $\varphi \in \Gamma$, if $\mathcal{M} \not\models \varphi$, then $\mathcal{N} \not\models \varphi$.

To demonstrate this proof strategy, but before introducing the additional complication of a forcing construction, Hirschfeldt presents two classic results: the conservativity of ACA_0 over PA, and of RCA_0 over $\mathrm{I}\Sigma_1$. He then turns to the foundational result in this area: Harrington's proof that WKL_0 is Π^1_1 conservative over RCA_0. The last part of chapter 7 proves, using Mathias forcing, that the principle COH introduced in chapter 6 is conservative over RCA_0 for all r-Π^1_2 sentences: those of the form $\forall X[\eta(X) \to \exists Y \rho(X,Y)]$ where η is arithmetic and ρ is Σ^0_3. This result gives a uniform proof that COH does not imply, over RCA_0, any principle that can be expressed as an r-Π^1_2 sentence, including Weak König's Lemma, RT^2_2, and SRT^2_2.

Given its importance in recent work in reverse mathematics, and the central role it plays in this book, it is natural to ask whether RT^2_2 corresponds to any particular subsystem of first order arithmetic. It has been known since Hirst [1987] that RT^2_2 implies $\mathrm{B}\Sigma^0_2$, and Chong, Slaman and Yang Chong et al. [2017] showed that it does not imply $\mathrm{I}\Sigma^0_2$. Hirschfeldt asks (7.14) if RT^2_2 Π^1_1-conservative over $\mathrm{B}\Sigma^0_2$. Patey and Yokoyama Patey and Yokoyama [2016] have recently shown that $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ is Π^0_3-conservative over $\mathrm{I}\Sigma_1$. $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ is thus Π^0_3 conservative over $\mathrm{B}\Sigma^0_2$ as well. It remains open whether this can be extended to full Π^1_1-conservativity, but Patey and Yokoyama's result shows that RT^2_2 has the same consistency strength as $\mathrm{I}\Sigma_1$, and moreover that adding it to WKL_0 does not result in an increase in consistency strength.

The results of the previous chapters are summarised in chapter 8 through a series of diagrams. Then, in chapter 9, Hirschfeldt turns to the study of principles weaker than RT^2_2, which is a major focus of current research. These principles include some we have already seen such as COH, but also many others including statements concerning linear and partial orders, combinatorial principles such as further weakenings of Ramsey's theorem, and statements from model theory such as the Atomic Model Theorem. Finally, chapter 10 presents a selection of further topics, including two sections on linear orders and one on systems stronger than ACA_0. Although a short chapter on such diverse topics can only provide a brief introduction to these other areas of reverse mathematics, it is nonetheless a useful addition that could open up the breadth of the subject to a newcomer. All in all, Hirschfeldt's decision to keep the focus of the book squarely on Ramsey's theorem and related topics is entirely justified: the clear narrative and the thoroughness with which the central ideas are treated makes *Slicing the Truth* an excellent book which this reviewer is happy to recommend.

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