Reverse Mathematics: A Philosophical Account

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Abstract

Beginning with the work of Friedman [1975, 1976], the programme of *reverse mathematics* has shown which axioms are required in order to prove many of the core theorems of ordinary mathematics. In the course of doing so, a striking phenomenon has emerged: almost every such theorem is either provable in a weak base theory corresponding to computable mathematics, or equivalent over that base theory to one of just four subsystems of second order arithmetic.

Although these results are clearly important, their philosophical ramifications are unclear. The prevailing view in the reverse mathematics community is that they reveal the strength of set existence principles required to prove given theorems of ordinary mathematics. Despite the prima facie plausibility of this view, the key concept of a set existence principle is left undefined and unanalysed. Moreover, the position depends heavily on the assumption that the coded representations of ordinary mathematical objects in second order arithmetic are semantically faithful, and thus that the formal counterparts of theorems of ordinary mathematics preserve the mathematical content of those theorems. Finally, while close connections have been drawn between results in reverse mathematics and important existing programmes in the foundations of mathematics, these connections and the possible role of reverse mathematics in answering foundational questions have not been the subject of a comprehensive philosophical enquiry.

This thesis tackles all three issues. In chapter 2, I analyse the concept of a set existence principle; argue for several constraints which any theory of this concept should satisfy; and argue for a novel interpretation on which set existence principles are understood as logically natural closure conditions on the powerset of the natural numbers. I then turn to representational issues: in chapter 3, I survey results in higher order reverse mathematics that demonstrate that some common codings are problematic, and draw some philosophical morals. Foundational questions are explicitly addressed in chapter 4, in which I examine how reverse mathematics can aid foundational inquiries, and the extent to which major subsystems of second order arithmetic correspond to existing foundations for mathematics. Finally, in chapter 5, I examine an alternative approach to reverse mathematics developed by Richard Shore. Building on the understanding obtained in the preceding chapter, I show that Shore's framework is not an appropriate one in which to address foundational questions.

To my family.

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Author's declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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Contents

List of Figures

1

Preliminaries

1.1 From axioms to theorems and back again

What axioms are truly necessary to prove particular theorems, or clusters of theorems? To answer this question, Harvey Friedman [1975, 1976] initiated a research programme called *reverse mathematics*. By formalising ordinary mathematical concepts and statements in the language of second order arithmetic, Friedman was able to show not only that many theorems of ordinary mathematics could be proved in relatively weak subsystems of second order arithmetic Z_2, but that such theorems often turned out to be equivalent (modulo a weak base theory) to the axioms used to prove them. A classic example is the least upper bound principle for the real numbers, which is equivalent not only to numerous different formulations of the principle but also to the axiom scheme of arithmetical comprehension.

The conception of the "ordinary mathematics" studied in reverse mathematics is somewhat imprecisely drawn. Simpson [2009] distinguishes between two parts of mathematics. On the one hand there is *set-theoretic mathematics*, that body of mathematical knowledge in which set-theoretic methods and concepts are inextricably embedded. This includes the more abstract forms of point-set topology and functional analysis, as well as set theory itself. On the other hand there is the subject matter of reverse mathematics, what Simpson calls *ordinary* or *non-set-theoretic mathematics*. These branches of mathematics are in some way independent of set-theoretic principles, and include real and complex analysis, geometry, countable algebra, number theory and combinatorics.

To prove a result in reverse mathematics, we start with a weak base theory B, and a formalisation τ of a given ordinary mathematical theorem. Assuming that τ is not provable in B—typically shown using a model construction—we use some stronger theory S (that extends B) to prove τ, often by a straightforward formalisation of the usual proof of that theorem. This gives us the forward direction of the equivalence. We then add the theorem τ to the base theory B and use the resulting theory $B+\tau$ to prove the axioms of the stronger system S. This "reversal" demonstrates that S and τ are equivalent, modulo the base theory B, and thus that the axioms of S are necessary in order to prove τ.

The formal framework used in reverse mathematics is second order arithmetic, a first-order logical system with two sorts of terms: natural number terms and set terms, which in the intended interpretation range over the natural numbers $\mathbb{N}$ and their powerset $\mathcal{P}(\mathbb{N})$. It has a long history in work on the foundations of mathematics; the most substantive classical developments relevant to reverse mathematics are those of Hilbert and Bernays [1968, 1970], and the system later appeared in works of the Polish school in connection with infinitary logic.[1]

Although far more expressive than the familiar systems of first-order arithmetic such as Peano arithmetic PA and its subsystems, second order arithmetic is still restricted in its expressive power. It cannot, for example, quantify over arbitrary sets of real numbers. This has some ramifications when we formalise ordinary mathematical properties and statements within second order arithmetic. The version of the least upper bound principle one can find in most analysis textbooks states that every *set* of real numbers with an upper bound has a least upper bound.[2] In contrast, the version studied in reverse mathematics states that every bounded *sequence* of real numbers has a least upper bound, since every countable sequence of real numbers can be coded by a single real, and individual real numbers (or at any rate, sets of natural numbers representing them) can be quantified over in second order arithmetic—indeed, it is for this reason that second order arithmetic has historically often been referred to as the first-order theory of analysis.

It is a striking fact that almost all theorems of ordinary mathematics studied thus far are provable in—and often equivalent to—just five basic systems of second order arithmetic. These systems are collectively known as the Big Five: RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0. All of them include the basic number-theoretic axioms of PA^- (first-order Peano arithmetic PA, minus the induction scheme) and the axiom scheme of Σ^0_1 induction (with set parameters). Where

[1] For example in publications such as Grzegorczyk, Mostowski, and Ryll-Nardzewski [1958] and Mostowski [1961], albeit typically in the guise of a second-order functional calculus, which allows one to avoid some of the coding machinery that must be employed when only quantification over sets is permitted.

[2] For example p. 4 of the classic Rudin [1976].

they differ is in the different strengths of set existence principles that they incorporate.

The most fundamental subsystem of second order arithmetic is RCA_0. This system is named for the *recursive comprehension axiom*, which restricts the comprehension scheme to Δ^0_1 sets, i.e. those which define recursive (computable) sets. RCA_0's importance is due to the fact that it serves as the standard base theory for reverse mathematics: the vast majority of equivalences proved in the field are proved over RCA_0, since although the theory is rich enough to prove that many formalisations of ordinary mathematical notions are well-defined, it is limited to the computable world, and cannot show that non-computable objects exist. The degree to which ordinary mathematical practice implicitly appeals to such objects is one of the major themes of reverse mathematics.

Somewhat stronger than RCA_0 is the system WKL_0, which consists of the axioms of RCA_0 plus the principle called *weak König's lemma*. This is a restriction of the classical König's lemma to binary trees: every infinite tree $T \subseteq 2^{<\mathbb{N}}$ has an infinite path. WKL_0 is just strong enough to prove theorems that rely on some form of compactness, such as Gödel's completeness theorem or the Heine/Borel covering lemma.

The system ACA_0 will be more familiar to logicians at large, since it is essentially just the second order version of Peano arithmetic PA. Its axioms consist of those of RCA_0 plus a strengthened comprehension principle which states that all arithmetically-definable sets exist. It proves every instance of the first-order induction scheme, but no more: it is therefore conservative over PA for sentences in the first-order language of arithmetic. Since it can define the Turing jump operator, ACA_0 is able to carry out many constructions that are impossible in WKL_0, and as such it can prove much stronger compactness and completeness properties for the real numbers. This enables one to develop most of the usual theory of real and complex analysis, including the Bolzano/Weierstraß theorem.

ACA_0 is also strong enough to prove *arithmetical transfinite induction*: every countable wellordering admits proof by induction for arithmetical formulas. However, it cannot prove the corresponding principle of *arithmetical transfinite recursion*, that sets defined by iterating arithmetical comprehension along countable wellorderings exist. This is the defining axiom of the much stronger system ATR_0, which is equivalent to a number of theorems concerning ordinals, such as the statement CWO that any two countable wellorderings are comparable. This is also the point in the reverse mathematics hierarchy where analysis gives way to descriptive set theory: ATR_0 is equivalent to, amongst others, the perfect set theorem.

All of the systems mentioned so far can be justified on predicative grounds, but with Π^1_1-CA_0 we take the first steps into impredicativity. This system's defining axiom is the Π^1_1 comprehension axiom, which asserts that every set definable by a Π^1_1 formula exists. It is equivalent to the existence of the hyperjump of every set, as well as other theorems from descriptive set theory such as the Cantor/Bendixson theorem.

A note on terminology: I shall occasionally use the term "reversal" to mean not just the implication from a theorem τ to a subsystem of second order arithmetic S, but the equivalence $S \leftrightarrow \tau$. This is in line with my goal to explain the significance of these equivalences, for which the reversal is essential and the distinguishing characteristic of reverse mathematics (hence, of course, its name). It will always be clear from the context whether the equivalence or just the implication $\tau \Rightarrow S$ is intended. The term "reversal" implies that the implication $S \Rightarrow \tau$ is already known, so I will not characterise implications $\tau \Rightarrow T$ where T is proof-theoretically weaker than τ as "reversals"; instead I shall simply call them "implications".

1.2 A typical reversal

The Bolzano/Weierstraß theorem is a fundamental result in real analysis which states that every bounded sequence of real numbers contains a convergent subsequence. We can express the theorem in second order arithmetic by defining sequences of real numbers as follows.

Within RCA_0, a *sequence of real numbers* is a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that for each $n \in \mathbb{N}$ the function $(f)_n : \mathbb{N} \to \mathbb{Q}$ defined by $(f)_n(k) = f((k, n))$ is a real number. Such a sequence *converges to* x, $x = \lim_n x_n$, if

$$\forall \varepsilon > 0\ \exists n \forall i (|x - x_{n+i}| < \varepsilon).$$

A sequence is *convergent* if $\lim_n x_n$ exists.

Theorem 1.2.1 (Friedman)**.** *The Bolzano/Weierstraß theorem is equivalent over* RCA_0 *to the arithmetical comprehension scheme.*

Proof. The usual proof of the Bolzano/Weierstraß goes through in ACA_0, giving us the forward direction of the equivalence. To get the reversal we work in RCA_0 and assume the Bolzano/Weierstraß theorem. Arithmetical comprehension is equivalent to Σ^0_1 comprehension, so it suffices to prove for some arbitrary Σ^0_1 formula $\varphi(n)$ that $\{\, n \mid \varphi(n) \,\}$ exists.

Let $\varphi(n) \equiv \exists k \theta(k,n)$ be Σ^0_1, i.e. θ is Σ^0_0, and for each $k \in \mathbb{N}$ define

$$c_k = \sum \left\{ 2^{-n} : n < k \wedge (\exists m < k)\theta(m,n) \right\}.$$

Then $\langle c_k : k \in \mathbb{N} \rangle$ is an increasing sequence of rational numbers which is bounded above by k. This sequence exists by Δ^0_1 comprehension. By the Bolzano/Weierstraß theorem, $c = \lim_k c_k$ exists. So we have

$$\forall n(\varphi(n) \leftrightarrow \forall k(|c - c_k| < 2^{-n} \rightarrow (\exists m < k)\theta(m,n))).$$

This shows the equivalence of a Σ^0_1 formula with a Π^0_1 formula, so by Δ^0_1 comprehension we have that

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n)).$$

This proves Σ^0_1 comprehension and hence arithmetical comprehension. □

We shall see in more detail how this result relates to other, similar theorems in real and functional analysis in §1.3. The reader interested in attaining a broader understanding of the reverse mathematics programme before diving into the philosophical fare on offer here would be well served by two recent survey papers: Simpson [2010] and Shore [2010]. The latter is aimed more at logicians and connects the field to other areas of mathematical logic. The main textbook on reverse mathematics is Simpson's book *Subsystems of Second Order Arithmetic* [Simpson 2009], which we shall consult throughout the course of this thesis.

1.3 The landscape of reverse mathematics

One of the most remarkable features about reverse mathematics is that although theorems of a certain kind tend to huddle within the same equivalence class—as the various limit principles listed below do—each one of the major subsystems WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0 is equivalent to many theorems from quite different branches of mathematics. Theorems of analysis, algebra and combinatorics all turn out to be equivalent to one another when formalised in RCA_0. This speaks to the unity of mathematics, where an algebraic theorem such as the existence of unique algebraic closures of every countable field is equivalent to the existence of suprema for continuous real-valued functions, a result that lies squarely in the purview of analysis.

More surprisingly, the vast majority of theorems studied to date are either provable within the base theory RCA_0, or are equivalent over RCA_0 to one of the other members of the Big Five. Since there are infinitely many distinct subsystems of second order arithmetic, one might expect ordinary mathematical theorems to be a little more spread out than this. This phenomenon, which I will typically refer to as the *Big Five phenomenon*, will be central to many of our later discussions.

Theorem 1.3.1 (Friedman/Simpson). *The following are pairwise equivalent over* RCA_0*:*

1. ACA_0.
2. *The least upper bound principle: Every bounded sequence of real numbers has a least upper bound.*
3. *Every Cauchy sequence of real numbers is convergent.*
4. *The monotone convergence theorem: Every bounded increasing sequence of real numbers is convergent.*
5. *The Bolzano/Weierstraß theorem: Every bounded sequence of real numbers has a convergent subsequence.*
6. *In any compact metric space, every sequence of points has a convergent subsequence.*
7. *The Ascoli lemma: Every bounded equicontinuous sequence of real-valued continuous functions on a bounded interval has a uniformly convergent subsequence.*

As mentioned above, one of the early results in reverse mathematics was the equivalence between two basic completeness properties for the real numbers, the least upper bound principle and the Bolzano/Weierstraß theorem, and the axiom of arithmetical comprehension [Friedman 1975, p. 238]. Proofs of this equivalence, and the rest of the results that make up theorem 1.3.1 above can be found in §III.2 of Simpson [2009], who generalises the Bolzano/Weierstraß theorem from the real numbers $\mathbb{R}$ to any complete separable metric space. A further generalisation of this theorem, the Ascoli lemma, is an important result in functional analysis. As Simpson shows, these theorems all occur at exactly the same level in the reverse mathematics hierarchy, since they are all equivalent to ACA_0. This illustrates two important properties of the hierarchy.

The first is that generalisations of theorems often turn out to have precisely the same proof-theoretic strength—or equivalently, theorems are often equivalent to their special cases. To name but one example, the Heine/Borel covering lemma for compact metric spaces is equivalent to the special case for the closed unit interval $[0,1]$. An increase in generality can therefore be obtained without a corresponding increase in the strength of the axioms required to prove the theorem.

Secondly, these theorems are stable with respect to the reverse mathematics hierarchy under some degree of generalisation or specialisation. This property is known in the reverse mathematics literature as *robustness*, and many of the key systems in reverse mathematics exhibit it by remaining stable under perturbations of their axioms. For example, ACA_0 is a stable system: arithmetical comprehension is equivalent not only to Σ^0_1 comprehension, but also to similar principles such as the existence of the Turing jump operator, and the existence of ranges of one-to-one functions. All of the Big Five are robust, but thus far few other systems are thought to have this property [Montalbán 2011, p. 432].[3]

We shall now survey a few important and illustrative equivalences between members of the Big Five and theorems of ordinary mathematics. These theorems are drawn from many different areas of mathematics: as well as the examples from analysis already discussed, there are theorems from algebra, combinatorics, logic and descriptive set theory. A more complete list of important equivalences between ordinary mathematical theorems and members of the Big Five can be found in Simpson [2010, pp. 116–9].

The following theorems are equivalent over RCA_0 to WKL_0.

1. *The Heine/Borel covering theorem: Every covering of the closed unit interval* $[0,1]$ *by a sequence of open intervals has a finite subcovering.*

   Like weak König's lemma itself, this theorem from analysis is a compactness result, stating that the closed unit interval is a compact space. As mentioned above, this result generalises to compact metric spaces without an increase in proof-theoretic strength [Simpson 2009, §IV.1].

2. *Every continuous function on* $[0,1]$ *is bounded.*

3. *The separable Hahn/Banach theorem: If f is a bounded linear functional on a subspace of a separable Banach space such that $\|f\| \leq 1$, then f can be extended to a functional $\tilde{f}$ on the entire space where $\|\tilde{f}\| \leq 1$.*

   This is a striking example of how the restriction to countably representable objects—in this case, separable Banach spaces—can reduce the proof-theoretic strength of a theorem. The more usual general form of the Hahn/Banach theorem is not provable in ZF set theory, although it is a theorem of ZFC [Pincus 1974].

[3] Sanders [2012, §7] discusses robustness in reverse mathematics at some length.

4. *Every countable commutative ring has a prime ideal.*

5. *Every countable field has a unique algebraic closure.*

6. *Every countable formally real field is orderable.*

7. *Every countable formally real field has a real closure.*

8. *Brouwer's fixed point theorem: Every continuous function* $\phi : [0,1]^n \to [0,1]^n$ *has a fixed point.*

9. *Gödel's completeness theorem: Every consistent countable set of sentences in the predicate calculus has a countable model.*

   This result is a robust one: similar theorems such as the compactness theorem for predicate calculus, and the compactness and completeness theorems for propositional logic with countably many atomic formulas are also equivalent to WKL_0 [Simpson 2009, §IV.3]. The definition of countable model used in these theorems incorporates the full elementary diagram, since the usual approach via a recursive satisfaction predicate is not available in RCA_0.

We have already met a number of theorems from analysis which are equivalent over RCA_0 to ACA_0 (theorem 1.3.1). The following theorems come from algebra and combinatorics, and are also equivalent over RCA_0 to arithmetical comprehension.

1. *Every countable commutative ring has a maximal ideal.*

2. *Every countable vector space over* $\mathbb{Q}$ *has a basis.*

3. *Every countable field of characteristic zero has a transcendence basis.*

4. *Every countable abelian group has a unique divisible closure.*

5. *König's lemma: Every finitely branching infinite tree has a path.*

   König's lemma provides an example where a generalisation does in fact lead to an increase in proof-theoretic strength.

6. *Ramsey's theorem: Every colouring of* $[\mathbb{N}]^k$ *(for any fixed* $k \geq 3$*) has a homogenous set.*

   Ramsey's theorem for $k = 2$ is weaker than ACA_0, and has sparked a large body of research on combinatorial theorems lying outside the Big Five. A good reference for the current state of the art is Hirschfeldt [2014], chapters 6 and 9. I also discuss this theorem in §2.5.

The following theorems are equivalent over RCA_0 to ATR_0.

1. *Any two countable wellorderings are comparable.*

2. *Ulm's theorem: Any two countable reduced abelian p-groups with the same Ulm invariants are isomorphic.*

3. *The perfect set theorem: Every uncountable closed, or analytic, set has a perfect subset.*

4. *Lusin's separation theorem: Any two disjoint analytic sets can be separated by a Borel set.*

5. *Every open game in $\mathbb{N}^{\mathbb{N}}$ is determined.*

6. *Every countable bipartite graph admits a König covering.*

   This theorem is from combinatorics, in particular the area known as matching theory. The reverse direction of this equivalence was proved by Aharoni, Magidor, and Shore [1992] and Simpson [1994] later proved the forward direction. The history of this result is discussed at length by Shore [2010, pp. 382–384].

The following theorems are equivalent over RCA_0 to Π^1_1-CA_0.

1. *Every countable abelian group is the direct sum of a divisible group and a reduced group.*

   Many of the statements equivalent to Π^1_1 comprehension are results in descriptive set theory; this result is strikingly different, since it hails not from an area with set-theoretic overtones, but from algebra.

2. *The Cantor/Bendixson Theorem: Every closed subset of $\mathbb{R}$ (or of any complete separable metric space) is the union of a countable set and a perfect set.*

   This is a classic result in descriptive set theory. It is typically proved using a tree representation for closed sets and a wellfoundedness argument. Its equivalence to Π^1_1 comprehension shows that this method of proof is in some sense ineliminable.

3. *Silver's Theorem: For every Borel (or coanalytic, or F_σ) equivalence relation with uncountably many equivalence classes, there exists a perfect set of inequivalent elements.*

This list of theorems, while illustrating the breadth of reverse mathematical results and the extent of the Big Five phenomenon, does not capture the full richness of the hierarchy of systems studied in reverse mathematics. A fuller picture will emerge in the course of this thesis, particularly in section 2.5 which discusses some combinatorial principles between ACA_0 and RCA_0 that behave in a more unruly manner than the Big Five.

1.4 Historical perspectives

As Sieg [1990, p. 872] points out, the importance of equivalences between major theorems of analysis and its basic principles were already apparent to Dedekind.[4] The basic principle is question is Dedekind's formulation of the *principle of continuity*: Given any partition of the real numbers $\mathbb{R}$ into disjoint sets X and Y such that for all $x \in X$ and all $y \in Y$, $x < y$, there exists a unique real number z such that z is either the greatest element of X or the least element of Y.

The following excerpt from section VII of *Stetigkeit und irrationale Zahlen* [Dedekind 1872] is quoted from the English translation [Dedekind 1901], and appears on pages 24–7 of the reprinted version [Dedekind 1963]. While his quoted remarks refer to "infinitesimal analysis", Dedekind's understanding of continuity was very close to our modern one, and he uses the term merely to refer to what we now call "real analysis".

> Here at the close we ought to explain the connection between the preceding investigations [of the foundations of analysis] and certain fundamental theorems of infinitesimal analysis.
>
> . . .
>
> One of the most important theorems may be stated in the following manner: "If a magnitude x grows continually but not beyond all limits it approaches a limiting value."
>
> . . .
>
> This theorem is equivalent to the principle of continuity, i.e., it loses its validity as soon as we assume a single real number not to be contained in the domain $\mathbb{R}$; or otherwise expressed: if this theorem is correct, then [the principle of continuity is also] correct.
>
> Another theorem of infinitesimal analysis, likewise equivalent to this, which is still oftener employed, may be stated as follows: "If

[4] I wish to thank Professor Sieg for bringing this part of Dedekind's work to my attention.

> in the variation of a magnitude x we can for every given positive magnitude δ assign a corresponding position from and after which x changes by less than δ then x approaches a limiting value."
>
> This converse of the easily demonstrated theorem that every variable magnitude which approaches a limiting value finally changes by less than any given positive magnitude can be derived as well from the preceding theorem as directly from the principle of continuity. ...
>
> These examples may suffice to bring out the connection between the principle of continuity and infinitesimal analysis.

Left undiscussed by Sieg is the extent to which Dedekind's investigation of these equivalences has been vindicated by results in reverse mathematics. The principle of continuity that Dedekind uses is not directly expressible in the language of second order arithmetic, since it quantifies over arbitrary sets of real numbers, but a similar completeness principle for the reals is so expressible, namely the sequential least upper bound principle: Every bounded sequence of real numbers has a least upper bound.

As we saw at the start of the previous section, the sequential least upper bound principle is equivalent over RCA_0 to a number of key theorems in analysis. Amongst them was the statement that every bounded increasing sequence of real numbers is convergent. This theorem should be familiar from Dedekind's quoted remarks: as he puts it, "If a magnitude x grows continually but not beyond all limits it approaches a limiting value." Dedekind also draws our attention to the theorem "If in the variation of a magnitude x we can for every given positive magnitude δ assign a corresponding position from and after which x changes by less than δ then x approaches a limiting value." This is just the principle that every Cauchy sequence of real numbers is convergent, and it is likewise equivalent to ACA_0. These reverse mathematical results demonstrate that the equivalences that concerned Dedekind are non-trivial, that is, these theorems are equivalent to one another over the weak base theory RCA_0, but they are not themselves entailed by that theory.

The development of mathematics within second order arithmetic can also be traced back to Dedekind, but for more substantive classical developments we must look beyond the end of the nineteenth century and into the early twentieth. A version of second order arithmetic, augmented with the full choice scheme and thus strictly stronger than Z_2 (albeit conservative for Π^1_4 formulas), was first introduced by David Hilbert and Paul Bernays in their *Grundlagen Der Mathematik* [Hilbert and Bernays 1968, 1970]. Their formalisations of analysis can be found in Supplement IV of the *Grundlagen*.

Reverse mathematics as a coherent programme was begun by Friedman [1975], who articulated the fundamental question that it investigates (although the term "reverse mathematics" was not used to describe it until much later):

> What are the proper axioms to use in carrying out proofs of particular theorems, or bodies of theorems, in mathematics? What are those formal systems which isolate the essential principles needed to prove them?

He then points to the phenomenon that now gives "reverse mathematics" its name [Friedman 1975, p. 235]:

> When the theorem is proved from the right axioms, the axioms can be proved from the theorem.
>
> When this theme applies, we have a unique formalization of the theorem, up to provable equivalence. [This] occurs surprisingly often, but not always.

In this first paper, Friedman studied subsystems of second order arithmetic with full induction. The restriction to Σ^0_1 induction was introduced in Friedman [1976], and is a feature of reverse mathematical research to the present day.

Contemporary reverse mathematical research is substantially shaped by two traditions or schools of thought: the *foundational tradition* and the *computational tradition*. Although they have many figures and approaches in common, these two traditions have distinctive motivations that colour the kind of research being done under the rubric of "reverse mathematics". The foundational tradition follows the line of research laid down by Friedman and advanced by Stephen Simpson and his students, namely studying the formalisation of theorems and indeed whole subfields from "ordinary" or "core" mathematics within second order arithmetic, and proving their equivalence to one or other of the major subsystems thereof.

Research in the computational tradition treats reverse mathematics more as a branch of applied computability theory. Techniques such as priority arguments and forcing which have developed in the context of questions concerning the structure of the Turing degrees, hyperarithmetical theory and so on, are best applied not to proving equivalences but to constructing models that satisfy one principle but not another—in other words, proving nonimplications. For example, to show that WKL_0 does not imply ACA_0, one can use the low basis theorem to construct a model M of WKL_0 in which all sets are low, and hence it does not contain the halting set $0'$. Since $0'$ can be proved to exist in

ACA_0, M is not a model of ACA_0. It is this tradition in reverse mathematical research that has led to focus on Ramsey's theorem for pairs and the intricate lattice of subsystems whose defining axioms stem from combinatorics, model theory and computability theory, and which are collectively known as the *reverse mathematics zoo* [Dzhafarov 2015]. Many of the major figures in this tradition are primarily recursion theorists, such as Denis Hirschfeldt, Richard Shore and Theodore Slaman.

1.5 Second order arithmetic and its subsystems

This section and the ones that follow are intended to bring the reader up to speed with the essentials of second order arithmetic, its major subsystems, and the general technical underpinnings of reverse mathematics. It is far from comprehensive and the reader interested in the mathematics for its own sake is advised to consult Simpson [2009], the primary textbook of the field.

Second order arithmetic is an extension of more familiar systems of arithmetic, such as first-order Peano arithmetic (PA) and its subsystems. In the intended interpretation, variables in first order arithmetic range over the natural numbers. Second order arithmetic also has such variables, called *number variables*, but in addition it has *set variables* which range over sets of numbers.

In this respect second order arithmetic is similar to second order Peano arithmetic, PA^2, which is a theory formulated in second-order logic. However, the semantics of Z_2 are importantly different from PA^2. With the full second order semantics, monadic second order predicates (akin to the set variables just mentioned) in PA^2 range over the entire powerset of the first order domain. As shown by Dedekind's categoricity theorem, this suffices to fix the interpretation of PA^2 to a single model (up to isomorphism), namely the standard model ω. Second order arithmetic and its subsystems, on the other hand, are formulated in first-order logic, and thus have many nonstandard models for both their first-order and second-order parts.

In this thesis we shall follow the convention used by Simpson [2009] and use the symbol $\mathbb{N}$ to refer to the "internal" natural numbers of theories in second-order arithmetic. In a model-theoretic context $\mathbb{N}$ refers to the natural numbers of the ambient model, in other words, whatever the range of the first-order variables happens to be. The symbol ω is reserved for the "real" or "external" natural numbers of a theory, which can be thought of as a set-theoretic construction or simply the natural numbers of the metatheory, regardless of what that metatheory is.[5]

The language of second order arithmetic L_2 is a two-sorted first order language with number variables $x_1, x_2, \ldots$ and set variables $X_1, X_2, \ldots$. Following the usual practice we abbreviate number variables with lowercase letters x, y, z, m, n, i, j, k and set variables with uppercase letters X, Y, Z. Other symbols are also employed; whether a symbol represents a first order or second order variable is always clear from context. The language of second order arithmetic has the signature

$$\mathrm{L}_2 = \{0, 1, +, \times, <, \in\}\,. \tag{1.1}$$

These are the constant symbols 0 and 1, binary function symbols $+$ and $\times$, and binary relation symbols $<$ and $\in$. L_2-structures are tuples of the form

$$M = \langle |M|, S, 0^M, 1^M, +^M, \times^M, <^M \rangle \tag{1.2}$$

where M is the domain of the first order variables and $S \subseteq \mathcal{P}(M)$ is the domain of the second order variables. 0^M and 1^M are elements of M, $+^M$ and $\times^M$ are binary operations on M, and $<^M$ is a binary relation on M. Set membership is interpreted as follows: $x \in Y$ iff $x^M \in Y^M$.

The full theory of second order arithmetic or Z_2 consists of three groups of axioms: the *number-theoretic axioms*;[6] the *induction axiom*; and the *comprehension scheme*. We define each in turn. All of the subsystems of second order arithmetic considered in reverse mathematics are obtained by weakening this full system of second order arithmetic.

$$n + 1 \neq 0 \tag{1.3}$$

$$m + 1 = n + 1 \rightarrow m = n \tag{1.4}$$

$$m + 0 = m \tag{1.5}$$

$$m + (n + 1) = (m + n) + 1 \tag{1.6}$$

$$m \cdot 0 = 0 \tag{1.7}$$

$$m \cdot (n + 1) = (m \cdot n) + m \tag{1.8}$$

$$\neg m < 0 \tag{1.9}$$

$$m < n + 1 \leftrightarrow (m < n \vee m = n) \tag{1.10}$$

Then there is the standard second order induction axiom, allowing induction over those sets the theory can prove exists.

$$\forall X((0 \in X \wedge \forall n(n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n(n \in X)). \tag{1.11}$$

[5] Although we shall also use it in a couple of other—hopefully not too confusing—ways.

[6] Simpson [2009] calls these the *basic axioms*.

Finally there is the full Z_2 comprehension scheme, Π^1_∞-CA, which asserts that every set defined by an L_2-formula φ exists.

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n)) \tag{1.12}$$

for all L_2-formulae φ with X not free. Both number and set parameters from the model are permitted in φ. This also holds for the restricted comprehension schemes such as arithmetical comprehension.

Theorem 1.5.1. Z_2 *proves the full second order induction scheme: the universal closures of*

$$(\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n\, \varphi(n) \tag{1.13}$$

for all L_2*-formulae* $\varphi(n)$.

Proof. Let X be the set such that $\varphi(n)$ holds for all n, where φ is an L_2-formula. X exists by the comprehension scheme 1.12 for φ, so we can just replace it by its defining condition in the induction axiom 1.11. □

Definition 1.5.2 (Σ^0_n induction scheme)**.** For each $n \in \omega$, the Σ^0_n *induction scheme*, in symbols Σ^0_n-IND, consists of the universal closures of all sentences of the form

$$(\varphi(0) \wedge \forall m(\varphi(m) \rightarrow \varphi(m+1))) \rightarrow \forall m(\varphi(m))$$

where $\varphi(m)$ is a Σ^0_n formula (possibly with free variables) of the language of second order arithmetic.

Where it simplifies presentation, the full induction scheme is abbreviated Σ^1_∞-IND. Systems with restricted induction axioms that cannot prove all instances of Σ^1_∞-IND are indicated by a subscripted '0'. These systems always have a counterpart system which *does* prove the full induction scheme. So for instance ACA_0 is the system defined by the arithmetical comprehension axiom and the Σ^0_1 induction scheme, while ACA consists of ACA_0 plus the full induction scheme Σ^1_∞-IND.

We now turn from full second order arithmetic Z_2 to its subsystems. A *subsystem* T *of* Z_2 is a formal system in the language L_2 such that each axiom φ of T is a theorem of Z_2. The following sections define the Big Five subsystems of second order arithmetic and explain some of their key properties.

1.6 Recursive comprehension

Definition 1.6.1 (recursive comprehension and RCA_0)**.** The axiom scheme of *recursive comprehension* consists of the universal closures of all sentences of the form

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is a Σ^0_1 formula and $\psi(n)$ is a Π^0_1 formula, possibly with free variables, and X is not free in $\varphi(n)$.

The axiom system RCA_0 is a formal theory in the language of L_2 consisting of the number-theoretic axioms; the Σ^0_1 induction axiom; and the recursive comprehension scheme.

Given a subsystem T of Z_2, we say that its *first-order part* is the set of sentences φ in the first-order language of arithmetic L_1 such that $T \vdash \varphi$. The first-order part of RCA_0 is just $I\Sigma_1$, the subtheory of Peano arithmetic where the induction scheme is restricted to Σ^0_1 formulas. Because consistency statements are Π^0_1, this means that RCA_0 is equiconsistent with $I\Sigma_1$ and has the same proof-theoretic ordinal, ω^ω. That this ordinal is wellfounded is equivalent over RCA_0 to the Hilbert basis theorem, as proved by Simpson [1988b]. This result is discussed in more detail in §2.7.

Like the other subsystems of Z_2 generally studied in reverse mathematics, RCA_0 is finitely axiomatisable. This follows from the representability in arithmetic of universal Turing machines via Kleene's T predicate. For consistency with the reverse mathematics literature we follow Simpson [2009, p. 244]'s terminology. Let

$$\pi(e, m_1, \ldots, m_i, X_1, \ldots, X_j)$$

be a Π^0_1 formula with precisely the displayed free variables. π is a *universal lightface* Π^0_1 *formula* if for all Π^0_1 formulas φ, RCA_0 proves that

$$\begin{aligned}\forall e \exists e' \forall m_1, \ldots, m_i \forall X_1, \ldots, X_j (\pi(e', m_1, \ldots, m_i, X_1, \ldots, X_j) \\ \leftrightarrow \varphi(e, m_1, \ldots, m_i, X_1, \ldots, X_j)).\end{aligned}$$

Such formulas can be constructed for any fixed $i, j < \omega$, by a result analogous to the enumeration theorem in recursion theory.

We can then give a finite axiomatisation of RCA_0. Let π be a fixed universal lightface Π^0_1 formula. Then the axioms of RCA_0 can be taken to consist of the pairing axiom[7]

$$\forall X \forall Y \exists Z (Z = X \oplus Y),$$

recursive comprehension in the form

$$\forall m(\neg\pi(e_0, m, X) \leftrightarrow \pi(e_1, m, X)) \rightarrow \exists Y \forall m(m \in Y \leftrightarrow \pi(e_1, m, X))$$

[7] See definition 5.1.1 of the recursive join operator $X \oplus Y$.

and Σ^0_1 induction in the form

$$(\neg\pi(e,0,X) \wedge \forall m(\neg\pi(e,m,X) \rightarrow \neg\pi(e,m+1,X))) \rightarrow \forall m \neg\pi(e,m,X).$$

The need for the pairing axiom can be eliminated by simply allowing two set parameters in the statement of the recursive comprehension axiom.

Before moving on to the next member of the Big Five, we shall briefly consider a special class of L_2-structures, *ω-models*. Their first-order part consists of the standard natural numbers $\omega = \{0,1,2,\dots\}$ while their second-order part consists of a collection of sets $S \subseteq \mathcal{P}(\omega)$. We shall often identify an ω-model with its second-order part.

RCA_0 has a minimum ω-model, namely

$$\begin{aligned}\mathrm{REC} &= \{ X \subseteq \omega \mid X \leq_{\mathbf{T}} \emptyset \} \\ &= \{ X \subseteq \omega \mid X \text{ is recursive} \}.\end{aligned}$$

The ω-models of RCA_0 are precisely the *Turing ideals*: subsets of $\mathcal{P}(\omega)$ which are upwards closed under recursive joins and downwards closed under Turing reducibility. ω-models of RCA_0 play an important role in chapter 5.

1.7 Weak König's lemma

Definition 1.7.1 (weak König's lemma and WKL_0)**.** *Weak König's lemma* is the assertion that every infinite subtree of $2^{<\mathbb{N}}$ has an infinite path.

The axiom system WKL_0 is a formal theory in the language of L_2 consisting of the axioms of RCA_0 plus weak König's lemma.

The first-order part of WKL_0 is $I\Sigma_1$, just like RCA_0. In fact, WKL_0 is Π^1_1 conservative over RCA_0; this fact is alleged by Simpson to have implications for the foundational role of WKL_0, which is discussed in §4.3. WKL_0 is therefore equiconsistent with RCA_0, but the addition of weak König's lemma allows it to prove many more ordinary mathematical theorems than RCA_0 can. There are many interesting results concerning the ω-models of WKL_0, not least that the countable ω-models of WKL_0 are precisely the Scott sets [Scott 1962].

1.8 Arithmetical comprehension

Definition 1.8.1 (arithmetical comprehension and ACA_0)**.** An L_2-formula is called *arithmetical* if it contains no set quantifiers. The axiom scheme of *arithmetical comprehension* consists of the universal closures of all sentences of the form

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is an arithmetical formula with X not free, but possibly containing other free variables.

The axiom system ACA_0 consists of the number-theoretic axioms; the Σ^0_1 induction scheme; and the arithmetical comprehension scheme.[8]

The first-order part of ACA_0 is Peano arithmetic, so the two theories are equiconsistent and have the same proof-theoretic ordinal, ε_0.

The arithmetical comprehension scheme is equivalent to the Σ^0_1 comprehension scheme. This follows from the fact that all of the comprehension schemes studied in reverse mathematics admit parameters. For the non-trivial direction of the equivalence, we reason by induction in the metatheory and suppose for some $k \in \omega$ that we have used Σ^0_k comprehension to show that a set X exists. We can then use Σ^0_0 comprehension to obtain its complement $\overline{X} = \{ n \mid n \notin X \}$, which will be a Π^0_k set. Finally we apply Σ^0_1 comprehension to obtain a new set which is $\Sigma^0_1(\overline{X})$, i.e. Σ^0_{k+1}.

This fact, coupled with the existence of universal lightface Π^0_1 formulas (introduced in §1.6 on RCA_0), allows us to prove the finite axiomatisability of ACA_0, since we can replace the infinite scheme of Σ^0_1 comprehension by a single instance in which the formula is a universal one.

Like other systems defined by comprehension schemes, ACA_0 has a minimum ω-model, namely the model

$$\begin{aligned} \mathrm{ARITH} &= \left\{ X \subseteq \omega \;\middle|\; (\exists n \in \omega)\ X \leq_{\mathbf{T}} 0^{(n)} \right\} \\ &= \{ X \subseteq \omega \mid X \text{ is arithmetical} \}. \end{aligned}$$

We shall pursue the connection between comprehension schemes and the existence of minimum ω-models further in section 2.2.

ACA_0 has a couple of notable extensions called ACA_0' and ACA_0^+. To understand their relationship to ACA_0, note that arithmetical comprehension is equivalent over RCA_0 to the fact that for any set $X \subseteq \mathbb{N}$ and any metatheoretic $n \in \omega$, the n-th Turing jump of X exists. ACA_0' strengthens ACA_0 by replacing the external n with an internal one: it asserts that for any $X \subseteq \mathbb{N}$

[8] Here my presentation differs slightly from that of Simpson [2009], who defines ACA_0 in terms of the induction axiom rather than the Σ^0_1 induction scheme. This is for uniformity of presentation, as then all of the Big Five have the same induction principle, rather than making RCA_0 and WKL_0 exceptions. This modification is not a substantial one since the two formulations are proof-theoretically equivalent, as should be clear from the result below about the first-order part of ACA_0.

and any $n \in \mathbb{N}$, $X^{(n)}$ exists. ACA_0^+ is stronger still, and consists of ACA_0 plus the assertion that for any $X \subseteq \mathbb{N}$, the ω-jump $X^{(\omega)}$ exists (where ω denotes the order type of the natural numbers under their standard ordering). Both ACA_0' and ACA_0^+ are intermediate systems with axioms that are weakenings of arithmetical transfinite recursion, the principle that the Turing jump operator can be iterated along any countable wellordering.

1.9 Arithmetical transfinite recursion

The definition of arithmetical transfinite recursion is somewhat technical, so for a detailed discussion we refer the reader to §V.2 of Simpson [2009]. Intuitively, the definition states that all those sets exist which can be defined by iterating arithmetical comprehension (or equivalently, the Turing jump) along a wellordering. The axiom system ATR_0 consists of the axioms of ACA_0 plus the scheme of arithmetical transfinite recursion.

The proof-theoretic ordinal of ATR_0 is Γ_0, also known as the Feferman–Schütte ordinal or the first impredicative ordinal. This system is therefore closely connected to Feferman's programme of predicative reductionism, which is discussed in §4.4.

1.10 Π^1_1 comprehension

Definition 1.10.1 (Π^1_1 comprehension and Π^1_1-CA_0)**.** A formula φ is Π^1_1 if it has the form $\forall Y \psi$ where ψ is an arithmetical formula. The Π^1_1 *comprehension scheme* consists of the universal closures of all formulas of the form

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

where φ is Π^1_1 and X is not free in φ.

The system Π^1_1-CA_0 consists of the number-theoretic axioms; the Σ^0_1 induction scheme; and the Π^1_1 comprehension scheme.

Π^1_1-CA_0 is the strongest of the subsystems of second order arithmetic that typically appears in reverse mathematical results. It is an impredicative system, making essential use of quantification over all sets of natural numbers, and can thus prove the existence of typical impredicatively defined objects such as Kleene's $\mathcal{O}$, the set of codes for recursive ordinals.

The reverse mathematics of Π^1_1 comprehension is largely focused on descriptive set theory. One striking exception to this is the theorem concerning abelian groups mentioned in §1.3. This result is due to Friedman et al. [1983], using a construction of Feferman [1975b].

2

Set existence and closure

2.1 The standard view

The major discovery of reverse mathematics is that ordinary mathematical theorems concerning countable and countably-representable objects are, in the vast majority of cases studied to date, either provable in the base theory RCA_0 or are proof-theoretically equivalent to another of the Big Five. This is a robust and remarkable phenomenon. Simpson [2010, p. 115] estimates that "several hundreds [of theorems] at least" have been found that fall into these five equivalence classes. While there are a few outliers—a number of which will be discussed in the course of this chapter—it is important to emphasise that they are relatively rare compared to theorems which fall within the purview of the Big Five.

The Big Five phenomenon, as we shall call it, demands an answer to the question of *significance*: impressive as this phenomenon is, what metaphysical or epistemic import does it have? What is the significance of reversals?[9]

The standard view in the field of reverse mathematics is that the significance of reversals lies in their ability to demonstrate what set existence principles are required to prove theorems of ordinary mathematics. They show us, for example, that arithmetical comprehension is required to prove the Bolzano/Weierstraß theorem, but only weak König's lemma is needed to prove the Hahn/Banach theorem for separable Banach spaces. A view of this sort, in more or less the terms just used, is articulated by Simpson [2009, p. 2] as his "Main Question": "Which set existence axioms are needed to prove the

[9] My concern in this chapter is what we can learn from the existence of reversals, rather than what we can learn from particular proofs of them. This is not to discount the possibility that proofs can have explanatory value, as has been suggested by much of the literature in the philosophy of mathematical practice, including Mancosu [2001], Weber and Verhoeven [2002], Mancosu [2008], Avigad [2010], Frans and Weber [2014]. But such issues will not be addressed here.

theorems of ordinary, non-set-theoretic mathematics?" Similar sentiments can be found elsewhere.[10]

The virtues of the standard view are worth enumerating. To begin with, it is straightforward and intuitive: the hierarchy of proof-theoretic strength that we see in the Big Five is understood as giving a hierarchy of set existence principles of increasing strength. The standard view also ties together the metaphysics and epistemology of reverse mathematics in a satisfactory way: if we interpret the language of second order arithmetic in a direct, realist way as referring to natural numbers and sets thereof, then knowing which axioms are necessary to prove some theorem τ gives us detailed information about which sets of natural numbers exist. Many of the features of reversals which were noted above are encompassed by this view. For example, the degree of nonconstructivity of a theorem θ is given by the strength of the nonconstructive set existence principles required to prove it. Finally, it allows us to understand the various foundational approaches which can be legitimately formalised within the reverse mathematics framework as being differentiated (in terms of their consequences, rather than their justifications) by their commitment to set existence principles of differing strengths. All in all, it is a strikingly appealing view.

It does, however, suffer from a major weakness: the central concept of a set existence principle is left unanalysed, and thus the precise content of the view is highly unclear. The primary goal of this chapter is to provide the missing analysis and thereby give the content of (a particular interpretation of) the standard view.

Before proceeding further, let us distinguish two concerns; we shall come later to the question of how separate they actually are. The first is the significance of reversals as a general matter: what do the equivalences proved in reverse mathematics between theorems of ordinary mathematics and canonical subsystems of second order arithmetic tell us? The second is the significance of reversals to a particular system? Depending on the account one offers, the latter may simply follow from the former; or it may not.

Consider weak König's lemma, which states that every infinite binary tree has an infinite path through it. This is effectively a compactness principle: one way of thinking of it is as stating that the Cantor space $2^{\mathbb{N}}$ is compact. One theorem of ordinary mathematics that is equivalent to WKL_0 is the Heine/Borel

[10] Such as in Friedman et al. [1983, p. 141], Brown and Simpson [1986, p. 557], Brown et al. [2002, p. 191], Avigad and Simic [2006, p. 139] and Dorais et al. [2015, p. 2]. There are many more examples to be found in the reverse mathematics literature, although it should be noted that many of the participants are students or coauthors of Simpson and thus the similarity in language is not surprising.

covering theorem, which states that every covering of the closed unit interval $[0, 1]$ by a sequence of open intervals has a finite subcovering. In other words, the Heine/Borel theorem states that $[0, 1]$ is compact. When put in these terms, it is not surprising that these two theorems are equivalent.

On the standard view, the significance of this equivalence is that it shows the set existence principle weak König's lemma to be needed in order to prove the Heine/Borel theorem. Regardless of what method of proof is used, a correct proof of this result will always appeal to some principle with the same proof-theoretic strength as WKL_0. Here we have deduced an account of the significance of a reversal from an account of the significance of reversals in general, namely the standard view that reversals demonstrate the strength of set existence axioms required to prove theorems of ordinary mathematics.

2.2 Set existence as comprehension

The standard view is that the significance of reversals is to be found in their calibration of the strength of ordinary mathematical theorems by demonstrating equivalences with set existence principles. In addition to this general thesis, Dean and Walsh [2012][11] take proponents of the standard view to be committed to a specific claim about the nature of set existence principles, namely that they are identical with comprehension principles. Leaving aside the question of whether or not this accurately characterises the standard position, either as generally stated or as asserted by particular authors, let us clarify the precise content of this view, which I call *set existence as comprehension* or SEC.

A comprehension scheme consists of the universal closures of all formulas of the form

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n)) \tag{2.1}$$

where φ belongs to some syntactically-defined set of formulas Γ. Common examples in reverse mathematics are recursive comprehension, arithmetical comprehension and Π^1_1 comprehension. These define the subsystems RCA_0, ACA_0 and Π^1_1-CA_0. Arithmetical comprehension allows φ to be any arithmetical formula: that is, a formula containing no set quantifiers. Π^1_1 comprehension states

[11] Unfortunately as of the time of writing Walter Dean and Sean Walsh's work on reverse mathematics remains unpublished. I base my attribution of views and arguments to Dean and Walsh on the slides of their talk [Dean and Walsh 2012]; on my memory of the talk as presented at the conference *The Limits and Scope of Mathematical Knowledge* in Bristol on March 18, 2012; and on my conversations and correspondence with both authors since that date. I present their argument that the standard view (or the "received view" as they term it) is mistaken in full below, since it is not yet available in published form.

that all sets definable by Π^1_1 formulas exist. Recursive comprehension states that the Δ^0_1 or recursive sets exist: in the above scheme, φ is a Σ^0_1 formula defining a set X such that there is a Π^0_1 formula φ' which also defines X.

One might worry that this view leads to too narrow a conception of what set existence principles are: perhaps there are comprehension principles that are defined by more fine-grained syntactic restrictions than the arithmetical and analytical hierarchy can provide. We therefore take a more general view, and take a *comprehension principle* to be the comprehension scheme associated with any subset Φ (resp. pair of subsets (Φ_Σ, Φ_Π) for Δ classes) of the formulas of L_2, just so long as for every class of formulas Γ (resp. pair of classes $(\Gamma_\Sigma, \Gamma_\Pi)$) in the arithmetical or analytical hierarchies, if every instance of Φ is provable from Γ-CA, then $\Phi \subseteq \Gamma$ (resp. $\Phi_\Sigma \subseteq \Gamma_\Sigma$ and $\Phi_\Pi \subseteq \Gamma_\Pi$). With this definition in hand we can state SEC with greater precision: set existence principles are just comprehension principles as we have defined them, and the significance of reversals lies in the strength of comprehension principles that ordinary mathematical theorems reverse to.

This view has a high degree of prima facie plausibility, given the important role played by comprehension principles in the foundations of mathematics since Frege's ill-starred attempt to reduce mathematics to logic, through the Russell paradox and the various restricted forms of comprehension that were proposed in response to it. Comprehension schemes are in general excellent candidate axioms. The idea that any given formal property (i.e. one defined by a formula of a formal language properly applied to some domain) has an extension is a highly credible basic principle, so long as appropriate precautions are taken to avoid pathological instances. Second order arithmetic is a fragment of simple type theory and so these difficulties cannot occur.

Moreover, comprehension schemes fall into straightforward hierarchies, with increasingly strong comprehension principles being characterised by a broader class of admissible definitions for sets. This harmonises with the reverse mathematical discovery that some theorems are true even of the recursive sets, while others require arithmetical comprehension to find appropriate witnesses. Such gradations can also be seen as hierarchies of acceptability: if one denies that uncomputable sets exist then recursive comprehension forms a natural stopping point; if one repudiates impredicativity then arithmetical comprehension could be a good principle to adopt.

Of the Big Five subsystems of second order arithmetic which are of primary importance to reverse mathematics, only three are characterised by comprehension schemes: RCA_0, ACA_0 and Π^1_1-CA_0. The intermediate systems WKL_0 and

ATR_0 are obtained by adding further principles to a comprehension scheme. In the case of WKL_0 we add *weak König's lemma*—the statement that every infinite binary tree has an infinite path through it—to the recursive comprehension scheme. For ATR_0 we add to arithmetical comprehension a further scheme of *arithmetical transfinite recursion* stating that those sets exist which can be defined by iterating the arithmetical operations along any wellordering. Nevertheless, one might well think that weak König's lemma and arithmetical transfinite recursion are stated in the form they are purely for instrumental reasons, and that they are in fact equivalent to comprehension schemes of some sort. The following result shows that this is not the case for weak König's lemma: although it is implied by the arithmetical comprehension scheme, it is not equivalent over RCA_0 to any subset of that scheme.[12]

Fact 2.2.1 (Dean/Walsh)**.** *No subset of the arithmetical comprehension scheme is equivalent over* RCA_0 *to weak König's lemma.*

Proof. The proof relies on the Simpson/Tanaka/Yamazaki theorem [2002]: If WKL_0 proves a statement of the form $\forall X \exists! Y \theta(X, Y)$ where θ is arithmetical, then so does RCA_0.

Assume for a contradiction that there is a set of arithmetical formulas Ψ such that RCA_0 proves that Ψ-CA is equivalent to weak König's lemma. By the finiteness of proof, we may assume without loss of generality that Ψ is finite. Then, since RCA_0 proves the existence of pairing functions, we may further assume that there a single instance of arithmetical comprehension

$$C_\varphi(X) \equiv \exists Y \forall n (n \in Y \leftrightarrow \varphi(n, X))$$

where φ has only the displayed free variables, such that

$$\mathsf{RCA}_0 \vdash \mathrm{WKL} \leftrightarrow \forall X C_\varphi(X).$$

We then define the arithmetical formula $\theta(X, Y) \equiv \forall n (n \in X \leftrightarrow \varphi(n, Y)$. Since identity for sets is defined as coextensionality, we then have that

$$\mathsf{RCA}_0 \vdash \forall X C_\varphi(X) \leftrightarrow \forall X \exists! Y \theta(X, Y).$$

By the Simpson/Tanaka/Yamazaki theorem, $\forall X \exists! X \theta(X, Y)$ is provable in RCA_0, so $\mathsf{RCA}_0 \vdash \mathrm{WKL}$, which is false, supplying our contradiction. $\square$

Fact 2.2.1 was pointed out by Dean and Walsh [2012], who argue that it shows weak König's lemma to be a counterexample to SEC. It is an open question whether or not weak König's lemma is equivalent over RCA_0 to some non-arithmetical instance of the full comprehension scheme. In order to support

[12] My thanks to Walter Dean and Sean Walsh for supplying the details of their proof, which I reproduce here, and making me aware of Simpson et al. [2002].

the claim that weak König's lemma is a counterexample to SEC, we therefore need a further argument. One such argument runs as follows: the syntactically defined complexity classes that give rise to comprehension schemes come with an associated ordering. Since it is this hierarchy of complexity classes that motivates the SEC view, the proponent of SEC is thereby committed to a constraint on which subsets of comprehension schemes constitute set existence principles. Namely, if a set existence principle is provable from a given comprehension scheme, it should be provably equivalent (over an appropriate weak base theory) to a subset of that scheme.

Crucially, WKL_0 is not merely a subsystem of second order arithmetic that is not equivalent to a comprehension scheme: it is a *mathematically natural* one, since weak König's lemma is equivalent over RCA_0 to many theorems of ordinary mathematics such as the Heine/Borel covering lemma, Brouwer's fixed point theorem, the separable Hahn/Banach theorem, and many other theorems of analysis and algebra.

Simpson [2010, p. 119] defines a subsystem of second order arithmetic as being mathematically natural just in case it is equivalent over a weak base theory to one or more "core" mathematical theorems. As the results summarised in chapter 1 show, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0 are mathematically natural systems, since each one is equivalent over RCA_0 to many theorems from different areas of ordinary mathematics.

The notion of mathematical naturalness appears to give us a partial answer to the question of the significance of reversals: by proving an equivalence between a theorem of ordinary mathematics τ and a subsystem of second order arithmetic S_τ, we thereby demonstrate that S_τ is a mathematically natural system. However, this still leaves us in the dark about the significance of the reversal for the theorem τ: what important property of this theorem of ordinary mathematics do we come to know when we prove its equivalence over a weak base theory to S_τ, that we did not know before?

It is also worth remarking that mathematical naturalness is not an absolute notion: some systems may, in virtue of being equivalent to many core mathematical theorems, be more mathematically natural than those which are only equivalent to a few such theorems. When a claim of the form "S is a mathematically natural system" is used in an unqualified way in the rest of this chapter, it should be taken to mean that S meets the minimum requirement of being equivalent to at least one core mathematical theorem.

Dean and Walsh's argument that SEC fails runs as follows: since weak König's lemma is neither a comprehension principle, nor equivalent to one, it cannot be a set existence principle (as by SEC, set existence principles are just comprehension schemes). So the significance of reversals to weak König's lemma cannot lie in the comprehension scheme that is both necessary and sufficient to prove them, since there is no such scheme. Either the significance of reverse mathematics does not lie in the set existence principles which theorems reverse to, or the set existence as comprehension view is false. Proponents of the set existence view are thus standing on shaky ground. They must adopt a more sophisticated way of spelling out their core contention, or abandon the idea that the significance of reversals lies in set existence principles.

Although weak König's lemma is not equivalent to a comprehension principle, it is equivalent to another type of schematic principle, namely a *separation scheme*. The separation scheme for a class of formulas Γ holds that if two formulas $\varphi, \psi \in \Gamma$ have disjoint extensions, then there exists a set including the extension of φ and excluding the extension of ψ. Weak König's lemma is equivalent over RCA_0 to Σ^0_1 separation, while ATR_0 is equivalent to Σ^1_1 separation.[13]

Definition 2.2.2 (separation scheme)**.** Let Γ be a class of formulas, possibly with parameters. The Γ-separation scheme, Γ-SEP, consists of all axioms of the form

$$\forall n(\neg(\varphi(n) \land \psi(n))) \rightarrow \exists X \forall n((\varphi(n) \rightarrow n \in X) \land (\psi(n) \rightarrow n \notin X)),$$

where $\varphi, \psi \in \Gamma$.

One response to Dean and Walsh's argument is to endorse the following more expansive conception of set existence principles: both comprehension schemes and separation schemes are set existence principles. In line with our existing terminology, we call this proposal SECS. This new conception does solve the immediate problem, since each of the Big Five are equivalent over RCA_0 to either a comprehension scheme or a separation scheme. But although the SECS view accommodates weak König's lemma, and thus evades the counterexample that sinks SEC, it does so at the price of a seemingly ad hoc modification to the view.

2.3 Conceptual constraints

The arguments levelled against the SEC account and its variants tacitly appeal to different constraints which the concept of a set existence principle should

[13] Both these results are well-known. The former is lemma 2.6 of Simpson [1984], which is related to theorem 6.1 of Jockusch and Soare [1972]. The latter is also due to Simpson and was announced in Simpson [1987] and is proved as theorem V.5.1 in Simpson [2009].

satisfy, if it is to play a role in explaining the significance of reversals. I shall now attempt to make these constraints explicit, by presenting three conditions which any satisfactory account of the concept of a set existence principle should meet, together with some reasons to believe that these conditions are plausible. I shall then show how the SECS account meets two of the stated conditions, but fails to satisfy the third.

(1) Nontriviality. *Not every subsystem of second order arithmetic expresses a set existence principle.*

(2) Comprehensiveness. *There are no subsystems S of second order arithmetic which are equivalent to ordinary mathematical theorems and yet are not equivalent to a statement expressing a set existence principle.*

(3) Unity. *Set existence principles are conceptually unified.*

Consider some account of set existence principles A. Such an account should lend substance to the claim that the significance of reversals lies in the set existence principles necessary to prove theorems of ordinary mathematics. If A does not satisfy the nontriviality condition (1) then it cannot do this. There are many statements of second order arithmetic that prima facie are not set existence principles, so violating the nontriviality condition entails failing to provide a theory that is truly an account of set existence principles at all. Simple examples of this are arithmetical statements; a class of examples which is more problematic from the standpoint of the standard view is studied in §2.7.

On the other hand, if A does not meet the comprehensiveness condition (2) then it also fails to provide an account of the significance of reversals in general—although it might still account for the significance of reversals to particular systems. Since the claim of the standard view is that significance of reverse mathematical results lies in giving the set existence principles necessary for the truth of a given theorem, any counterexample reduces the plausibility of the claim in direct proportion to the mathematical naturalness of the system which A does not account for. This is why the fact that weak König's lemma is not a comprehension scheme is so damaging to the SEC account: since it has been proved equivalent to many tens of core mathematical theorems, we have far more reason to abandon the philosophical view that reversals track set existence principles, let alone the specific thesis that they track degrees of comprehension, than we do to think that the mathematical naturalness of weak König's lemma is some kind of mirage or formal artefact, which is what would be required if we sought to elude the conclusion that it really constitutes a counterexample to SEC.

Any failure of A to meet the unity condition (3) has a somewhat different character. The standard view is an attempt to provide a general account of the significance of reversals, one that does not make overt reference to particular systems. Such generality requires that the systems which A countenances as set existence principles have some features in common. For example, while recursive comprehension, arithmetical comprehension and Π^1_1 comprehension are all different, the SEC account still satisfies the unity condition precisely because they are all comprehension schemes, and it can offer a theory under which all comprehension schemes can legitimately be considered to be set existence principles. If A does not satisfy the unity condition then it cannot be considered as offering a satisfactory theory of set existence principles; if no account meeting this condition can be found then we are reduced to merely offering specific accounts of the significance of reversals to particular systems, rather than a general theory of the significance of reverse mathematical results.

Accounts of the concept of a set existence principle can satisfy the unity condition in stronger or weaker ways. When there is a strong connection between the different systems considered to be set existence principles, the account satisfies the unity condition to a greater degree. In such cases the significance of reversals to a particular system S will in large part be given in terms of the theory of set existence principles, rather than in terms of specific properties of S that are at substantial variance to other set existence principles. The SEC account exhibits this property: different comprehension schemes are clearly all very much the same type of principle, and can be obtained by simple syntactic restrictions on a stronger principle, namely the full comprehension scheme. Nevertheless, requiring that any theory of set existence principles satisfies the unity condition to the same degree that the SEC account does seems like an onerous requirement that may well be impossible to meet in a theory that also satisfies the comprehensiveness condition. Allowing for theories to satisfy the unity requirement to a lesser degree, and have different set existence principles bear a mere family resemblance to one another, rather than be strictly of the same type of axiom in some strong syntactic sense, seems like a reasonable relaxation of the condition.

2.4 A counterexample to SECS

Admitting separation schemes as set existence principles is, prima facie, an ad hoc modification of the SEC view that seems to weaken one of the main strengths of the SEC view, namely its strong satisfaction of the unity condition

(3). The primary point of difference between separation and comprehension schemes is that as straightforward definability axioms, comprehension schemes tell us which particular sets exist. Separation schemes, on the other hand, do not always do so: an axiom asserting the mere existence of a separating set may not pin down a particular set as the witness for this assertion.[14] This fact is an illustration of Friedman [1975]'s point that "Much more is needed to define explicitly a hard-to-define set of integers than merely to prove their existence."

To rebut the argument that SECS is ad hoc, and show that it does after all satisfy the unity requirement, we must show that there is some degree of conceptual commonality between comprehension schemes and separation schemes. Following Lee [2014], we can treat the Big Five in a unified way by understanding them as *interpolation schemes*. These hold that given two predicates $\varphi(n)$ and $\psi(n)$, if the extension of the latter is a superset of the extension of the former, then an interpolating set Z exists such that $\{ n \mid \varphi(n) \} \subseteq Z \subseteq \{ n \mid \psi(n) \}$.

Definition 2.4.1 (interpolation scheme)**.** Let Γ and Δ be sets of L_2-formulas, possibly with parameters. The Γ-Δ *interpolation scheme*, Γ-Δ-INT, is the set of all sentences of the form

$$\forall m(\varphi(m) \to \psi(m)) \to \exists X \forall m((\varphi(m) \to m \in X) \wedge (m \in X \to \psi(m))) \tag{2.2}$$

where $\varphi \in \Gamma$ and $\psi \in \Delta$.

As Lee [2014] points out, all of the Big Five are equivalent to interpolation schemes. RCA_0 is equivalent to Π^0_1-Σ^0_1-INT; WKL_0 to Σ^0_1-Π^0_1-INT; ACA_0 to Σ^0_1-Σ^0_1-INT; ATR_0 to Σ^1_1-Π^1_1-INT; and Π^1_1-CA_0 to Σ^1_1-Σ^1_1-INT. This should go at least some way towards ameliorating our worry that SECS fails to satisfy the unity condition (3), since we can now see that both comprehension schemes and separation schemes are actually interpolation schemes.

Mere syntactic unity should not by itself convince us of the conceptual unity of comprehension principles and separation principles; after all, a sufficiently broad syntactic class of sentences will eventually unify all statements. The notion of an interpolation scheme is, however, relatively narrow and it is not hard to see that it is a reasonably straightforward generalisation of the concepts of separation and comprehension. The comprehension scheme for some class of formulas Ξ can be derived from the Ξ-Ξ interpolation scheme, since for any instance of comprehension we can use the given formula in both places in the

[14] A striking theorem in this vein is that the only sets which every ω-model of Σ^0_1-separation (i.e. WKL_0) has in common are the recursive ones [Simpson 2009, corollary VIII.2.27].

interpolation scheme and thus derive the comprehension instance. Separation schemes, on the other hand, arise when given some formula class Δ, the formula class Γ consists of the negations of the formulas in Δ, such as when $\Delta = \Sigma^0_1$ and $\Gamma = \Pi^0_1$.[15]

Nonetheless, even if we grant that SECS satisfies the unity condition (3), it still fails to offer a satisfactory theory of set existence principles, since there is a mathematically natural counterexample which shows that it does not satisfy the comprehensiveness condition (2). That counterexample is the axiom known as *weak weak König's lemma.* Weak weak König's lemma was introduced by Yu [1987], and as the name suggests, it is a further weakening of weak König's lemma. Weak König's lemma asserts that every infinite binary tree has an infinite path; weak weak König's lemma is the restriction of this principle to binary trees with positive measure.

Definition 2.4.2 (weak weak König's lemma)**.** *Weak weak König's lemma* is the statement that if T is a subtree of $2^{<\mathbb{N}}$ with no infinite path, then

$$\lim_{n\to\infty} \frac{|\{\, \sigma \in T \mid \mathrm{lh}(\sigma) = n \,\}|}{2^n} = 0.$$

The system WWKL_0 is given by adjoining the axiom weak weak König's lemma to the axioms of RCA_0.

The system WWKL_0 obtained by adjoining weak weak König's lemma to RCA_0 is strictly intermediate between RCA_0 and WKL_0 [Yu and Simpson 1990], and is equivalent over RCA_0 to a number of theorems from measure theory, such as a formal version of the Vitali Covering Theorem [Brown et al. 2002]; the countable additivity of the Lebesgue measure [Yu and Simpson 1990]; and the monotone convergence theorem for the Lebesgue measure on the closed unit interval. A survey of results in this area is given in Simpson [2009, §X.1]. These equivalences show that weak weak König's lemma is mathematically natural, in the sense of Simpson. By the comprehensiveness condition (2) we should therefore expect a good account of set existence principles to include it.

It follows from fact 2.2.1 that WWKL_0 is not equivalent to any subset of the arithmetical comprehension scheme, since every model of WKL_0 is also a

[15] Separation schemes and comprehension schemes do not always dovetail as nicely as they do for the Big Five. One might expect, for example, that the Π^1_1-separation scheme would be equivalent to Δ^1_1-CA_0, but that is not the case: Montalbán [2008] showed via a forcing construction that Π^1_1-separation lies strictly between Δ^1_1-CA_0 and Σ^1_1-AC_0. However since ordinary mathematical theorems that are also theorems of hyperarithmetical analysis are few and far between—Montalbán [2006]'s example of a statement about indecomposable linear orderings is the only substantial example—this fact cannot at present be considered a problem for the SECS view.

model of WWKL_0. Yu and Simpson [1990, §2, pp. 172–3] proved that not every model of WWKL_0 is a model of WKL_0. Their argument involves the construction of what is known as a *random real model*, and it implies that WWKL_0 is not equivalent to a separation principle either. SECS therefore fails to accommodate a mathematically natural system, and so fails to satisfy the comprehensiveness condition (2).

Theorem 2.4.3 (Yu and Simpson 1990)**.** *Weak weak König's lemma is not equivalent over* RCA_0 *to any subscheme of the* Σ^0_1*-separation scheme.*

A virtue that it would be reasonable to expect of any account of set existence principles is the ability to incorporate the discovery of new subsystems of second order arithmetic which turn out to be equivalent to theorems of ordinary mathematics. Banking on SEC or its extensions amounts to a bet that all such new systems will be comprehension schemes or separation schemes. The discovery of weak weak König's lemma and the role it plays in the reverse mathematics of measure theory shows that such optimism is unfounded even for the systems which are already known. In the next section I will advance an account of set existence principles which does not suffer from this weakness.

2.5 Closure conditions

In a sense the term *set existence principles* is an unfortunate one, since it might suggest that the relevant principles assert the unconditional existence of some sets, independently of the other axioms of the theory. A better term, which more accurately captures the natures of these axioms, is *closure conditions*—more precisely, closure conditions on the powerset $\mathcal{P}(\mathbb{N})$ of the natural numbers. Weak König's lemma is a closure condition in this sense: it asserts that $\mathcal{P}(\mathbb{N})$ is closed under the taking of infinite paths through infinite binary trees.

This example shows that closure conditions are not, in general, bare or unconditional statements of set existence. Rather, they hold that given the existence of any object X with certain properties, there exists some other object Y with certain properties. Recursive comprehension proves the existence of infinite, recursive subtrees of $2^{<\mathbb{N}}$; weak König's lemma states that each such tree has an infinite path through it. In the absence of a suitable base theory such as RCA_0, weak König's lemma would not allow us to prove the existence of any sets at all. In this sense it is a conditional set existence principle.

Comprehension schemes, on the other hand, appear at first blush to be set existence principles *tout court*. Nevertheless, they too are better understood as closure conditions, because the comprehension principles used in reverse mathematics all admit parameters. Comparing the standard formulation of recursive comprehension (in which parameters are allowed) with the parameter-free version makes this clear.

The parameter-free recursive comprehension scheme asserts the existence of those sets definable in a Δ^0_1 way, without reference to any other sets. But recursive comprehension with parameters instead asserts that $\mathcal{P}(\mathbb{N})$ is closed under relative recursiveness: if $X, Y \subseteq \mathbb{N}$ exist, so does every $Z \leq_{\mathbf{T}} (X \oplus Y)$. It is easy to construct models of parameter-free recursive comprehension that are not models of RCA_0: REC $\cup \{ X \}$ will do, for any $X \subseteq \omega$ such that there is a $Y \subseteq \omega$ with $\emptyset <_{\mathbf{T}} Y <_{\mathbf{T}} X$. This model does not contain Y, since it is neither recursive nor equal to X. But any ω-model of RCA_0 containing X would also have to include Y, since the standard version of recursive comprehension asserts that the powerset is closed under Δ^0_1 definability with parameters, not merely that the sets definable without parameters in a Δ^0_1 exist—and since $Y <_{\mathbf{T}} X$, Y is Δ^0_1 definable from X. Similar points apply to arithmetical comprehension and Π^1_1 comprehension.

While comprehension principles do have a different flavour to other closure conditions, they can often be characterised in equivalent ways which more closely hew to the model described above for weak König's lemma. Arithmetical comprehension, for example, is equivalent over RCA_0 to König's lemma: every finitely branching infinite subtree of $\mathbb{N}^{<\mathbb{N}}$ has an infinite path through it. Π^1_1-CA_0 is equivalent over RCA_0 to the statement that for every subtree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, if T has an infinite path then it has a leftmost such path [Avigad and Simic 2006, lemma 3.3].

With these points in mind, the main thesis of this chapter is the following: the significance of a provable equivalence between a theorem of ordinary mathematics τ and a subsystem T of second order arithmetic lies in telling us what closure conditions $\mathcal{P}(\mathbb{N})$ must satisfy in order for τ to be true. This is a bit of a mouthful, so we shall adopt the following slogan as an abbreviation for the view: *reversals track closure conditions.*

I do not claim complete originality for this view. Feferman [1992, p. 451] identifies set existence principles with closure conditions in his discussion of what mathematical existence principles are justified by empirical science (via the indispensability argument). Similar positions have also been taken in the reverse mathematics literature, for example by Dorais, Dzhafarov, Hirst, Mileti, and Shafer [2015, p. 2], who write that each subsystem studied in reverse mathematics "corresponds to a natural closure point under logical, and more specifically, computability-theoretic, operations", and by Chong, Slaman, and Yang

[2014, p. 864], who write that "Ultimately, we are attempting to understand the relationships between closure properties of $2^{\mathbb{N}}$". Shore [2010] also states that the Big Five correspond to recursion-theoretic closure conditions. This is clearly something in the air. However, none of these authors make precise what they mean by a closure condition, nor draw out the consequences of this view, although Chong et al. certainly consider it to have consequences for the practice of reverse mathematics: for instance they take it to show that ω-models have a particular importance.

On this particular point, more will be said later. For now, let us attempt to clarify the content of the view that reversals track closure conditions. Towards that end let us draw a distinction between two things: a closure condition in itself, and the different axiomatisations of that closure condition. Closure conditions are extensional: they are relations which the powerset $\mathcal{P}(\mathbb{N})$ may be closed under. Axiomatisations of closure conditions are intensional: one and the same closure condition will admit of infinitely many different axiomatisations (or as they may be thought of, presentations). So for example, the Turing jump operator gives rise to a closure condition, of which some of the better-known axiomatisations are (modulo the base theory RCA_0): the arithmetical comprehension scheme; König's lemma; and the Bolzano/Weierstraß theorem.

The upshot of this distinction is that by proving reversals we show that different theorems of ordinary mathematics correspond to the same closure conditions. The significance of reversals thus lies, at least to a substantial extent, in placing these theorems in a hierarchy of well-understood closure conditions of known strength. Note also that there is a duality here: an equivalence between a theorem τ and a system S_τ tells us something about τ, namely its truth conditions in terms of what closure condition must hold for it to be true, but it also tells us something about the closure condition itself, namely how much of ordinary mathematics is true in $\mathcal{P}(\mathbb{N})$ when that closure condition holds.

The view that reversals track closure conditions has some marked advantages over the SEC account and its variants. Most notably, it can accommodate all of the counterexamples discussed so far. Weak König's lemma is clearly a closure condition. So is weak weak König's lemma, and thus the new account also succeeds where the SECS view fails. Other principles which have been studied in reverse mathematics—arithmetical transfinite recursion, choice schemes, and many others—can all be understood as expressing closure conditions on $\mathcal{P}(\mathbb{N})$. Moreover, this account will also accommodate any similar principle discovered to be equivalent to a theorem of ordinary mathematics.

The Big Five form a linear order under the relation of proof-theoretic

$$\begin{tikzcd}
\Pi^1_1\text{−}\mathsf{CA}_0 \arrow[d, Rightarrow] & & \\
\mathsf{ATR}_0 \arrow[d, Rightarrow] & & \\
\mathsf{ACA}_0 \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & & \\
\mathsf{WKL}_0 \arrow[d, Rightarrow] & \mathsf{RT}^2_2 \equiv \mathsf{SRT}^2_2 + \mathsf{COH} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
\mathsf{WWKL}_0 \arrow[d, Rightarrow] & \mathsf{SRT}^2_2 \arrow[dl, Rightarrow] & \mathsf{COH} \arrow[dll, Rightarrow] \\
\mathsf{RCA}_0 & &
\end{tikzcd}$$

Figure 2.1: Provability diagram for the Big Five, WWKL_0 and the Ramsey-theoretic systems discussed in this section. All arrows denote strict implications: $A \Rightarrow B$ means that RCA_0 proves that A implies B, but not conversely.

strength. Adding weak weak König's lemma does not change the picture: WWKL_0 is a stronger system than RCA_0, but weaker than WKL_0. The SEC view is in part appealing because syntactic complexity provides a simple way to generate a linear hierarchy of natural systems of increasing strength. As we have seen, this account is susceptible to counterexamples that consist of intermediate systems which are not equivalent to comprehension schemes. A different kind of problem is posed by incomparable statements, i.e. φ and ψ such that RCA_0 proves neither that $\varphi \rightarrow \psi$ nor that $\psi \rightarrow \varphi$. Such examples sit uneasily with an account such as SEC whose appeal seems to include the neat linear order of systems it provides, founded on an increase in the syntactic complexity of formulas allowed into the comprehension scheme.

Much recent research in reverse mathematics has focused on the "Reverse Mathematics Zoo"[16] of systems between ACA_0 and RCA_0, which form not a linear order but a rather messy directed graph. Thus far there are few examples of ordinary mathematical theorems which fall outside the Big Five, and thus few examples of *incomparable* statements. However, there is one striking example which has been extensively studied: Ramsey's theorem for pairs, or as it is usually known, RT^2_2.

[16] `http://rmzoo.math.uconn.edu` [Dzhafarov 2015]. More implication diagrams can be found in Shore [2010], pp. 391 and 394, and chapter 8 of Hirschfeldt [2014] on pp. 139–142.

Definition 2.5.1. Given a set $X \subseteq \mathbb{N}$ and any $n \in \mathbb{N}$, the set of finite subsets of X with size n is denoted $[X]^n$. A *k-colouring of* $[X]^n$ is a function $f : [X]^n \to \mathbb{N}{\restriction_k}$. A set $H \subseteq X$ is *homogeneous* for a k-colouring f of $[X]^n$ if f is constant on $[H]^n$, i.e. all n-element subsets of H are assigned the same colour by f.

Ramsey's theorem states that for all $k, n \in \mathbb{N}$, every k-colouring of $[\mathbb{N}]^n$ has an infinite homogeneous set. This statement implies ACA_0, and has a natural class of weakenings, where we simply fix k and n to be particular natural numbers. In particular the statement RT^n_k is that given some fixed $n, k \in \omega$, every k-colouring of $[\mathbb{N}]^n$ has an infinite homogeneous set. One particular instance of this scheme, *Ramsey's theorem for pairs* or RT^2_2, has been the object of intense study in reverse mathematics [Cholak et al. 2001]. It stands out as a rare example of a system which falls outside the usual linear order of the Big Five, by being incomparable with WKL_0. This result follows from the work of Jockusch [1972], who proved that WKL_0 does not imply RT^2_2, and Liu [2012], who proved that RT^2_2 does not imply WKL_0.

Ramsey's theorem for pairs is, given its relationship to other, similar combinatorial statements, clearly a reasonably natural combinatorial principle. One might, however, hesitate before anointing RT^2_2 a theorem of ordinary mathematics—at least as that term has been used so far. After all, it is a combinatorial statement whose main use has been in logic. Drawing a parallel with the Paris–Harrington statement might be fruitful. It too is a combinatorial principle related to Ramsey's theorem, and it too has been claimed as a natural example, in that case of a mathematically natural statement in the language of arithmetic which is unprovable in Peano arithmetic.

We can put aside this concern for now, and instead make a conditional claim. If RT^2_2 provides us with an example of an ordinary mathematical theorem that is incomparable with one of the Big Five (or if some new example of this phenomenon is discovered in the future), then the apparent linearity of systems provided by SEC and similar views will find it problematic to accommodate the more complex landscape induced by such examples. The view that set existence principles are closure conditions, on the other hand, seems much more amenable to the existence of incomparable systems. There is nothing in the notion of a closure condition—save where our intuitions have been corrupted by the expectation of linearity induced by staring at the Big Five for too long—that rules out the existence of incomparable closure conditions.

A related phenomenon is that of splitting theorems, which show that some system T can be 'split' into two seemingly simpler systems T_1 and T_2 whose conjunction is equivalent to T. The most prominent example of this is again provided by RT^2_2, which Cholak, Jockusch, and Slaman [2001, lemma 7.11] showed to be equivalent to the conjunction of two other principles: stable Ramsey's theorem for pairs, SRT^2_2, and the cohesiveness principle COH. A colouring $f : [\mathbb{N}]^2 \to \{ 0, 1 \}$ is *stable* if for every $x \in \mathbb{N}$ there exists a colour $c \in \{ 0, 1 \}$ such that for all sufficiently large y, $f(x, y) = c$.

SRT^2_2: Every stable 2-colouring of pairs of natural numbers has an infinite homogeneous set.

COH: For every sequence of sets $\langle A_k | k \in \mathbb{N} \rangle$ there exists an infinite set B such that, for every i, either $B - A_i$ or $B \cap A_i$ is finite.

Neither of these two principles imply one another over RCA_0: they are incomparable.[17] Yet their conjunction is equivalent to a third principle, namely RT^2_2. Again, the linear nature of comprehension principles seems to militate against the incorporation of splitting systems into the SEC account, at least without some substantial conceptual overhaul. On the other hand, that we can obtain new closure conditions by conjoining existing ones is a simple and appealing principle.

Along with systems defined by a conjunction of two principles, there are also disjunctive systems. Perhaps surprisingly, there are even reversals to such systems from theorems of ordinary mathematics, such as Friedman et al. [1993]'s proof that the existence for all n of n-fold iterates of continuous mappings of the closed unit interval into itself is equivalent to the disjunction of Σ^0_2 induction and weak König's lemma.

Neither of the two properties just discussed, namely the existence of incomparable systems and splitting systems, are supposed to provide knock-down arguments against SEC or its variants. The point is rather to show that if we take set existence principles to be closure conditions, it gives us a supple framework which can accommodate these interesting features of the reverse mathematics hierarchy. That it does so should give us confidence that future discoveries of theorems of ordinary mathematics with such features can be incorporated into the view without the ad hoc modifications that the SEC account seems to require.

Before moving on, I must stress that I have not attempted to provide a definition of a closure condition. Instead I have given an intuitive account, and argued informally that at least the systems listed above, including both the Big Five and the major counterexamples to the SEC view and its variants, are in

[17] Cholak, Jockusch, and Slaman [2001] showed that COH does not imply RT^2_2, and thus does not imply SRT^2_2. The converse was recently proved by Chong, Slaman, and Yang [2014].

fact axiomatisations of closure conditions. One property that all these systems have in common is that they are proper extensions of RCA_0—that is to say, their axioms are not all provable in RCA_0. Moreover, the equivalences which I take to show that different statements express the same closure condition are in general proved in the usual base theory for reverse mathematics, namely RCA_0. One might therefore wonder about the degree to which the concept of a closure condition on $\mathcal{P}(\mathbb{N})$ is relative to the base theory: are there RCA_0-closure conditions, ACA_0-closure conditions and so on? If so, what are the implications for the account? These concerns will be taken up in §2.7.

The view that reversals track closure conditions is intended as a reconstruction of the standard view: reversals are significant because they tell us what set existence axioms are necessary to prove theorems of ordinary mathematics. As such, it is an attempt to give a relatively precise and well-motivated version of the set existence view, while accommodating counterexamples to other versions of the view such as SEC. However, if one hews to Dean and Walsh's claim that advocates of the standard view are committed to SEC, then the present account must instead be understood as a new theory about the significance of reversals, rather than as a way of spelling out the standard view. Little hangs on this exegetical detail; the key question is whether or not the view that reversals track closure conditions provides a satisfactory account of the epistemic and metaphysical significance of reversals.

To determine the answer to this question, we return to the three conditions that, as I argued in §2.2, any account of set existence principles should satisfy: nontriviality (1), comprehensiveness (2) and unity (3). By analysing the notion of a set existence principle in terms of closure conditions on the powerset of the natural numbers, the account clearly offers a unified picture of what set existence principles are. However, the notion of a closure condition is a very general one. In fact, there do not seem to be any prima facie reasons to rule out (at the very least) every Π^1_2 sentence expressing a closure condition. By way of contrast, the specificity of the concept of a comprehension principle means that the SEC account strongly satisfies the unity condition. But this very feature undermines its suitability as an analysis of the concept of a set existence principle, since it fails to be sufficiently comprehensive, as the existence of striking counterexamples such as weak König's lemma illustrates.

We therefore must conclude that although the view that reversals track closure conditions satisfies the unity condition, it only weakly satisfies it. As such, it is reasonable to wonder to what degree the view can offer a compelling explanation of the significance of reversals, since if it is easy for a relation to be considered a closure condition, then it is unclear what light this can shed on the importance of particular reversals. To put it another way, what more does this account say about the difference between a theorem τ's being equivalent to weak König's lemma rather than arithmetical comprehension?

Note that whilst being a closure condition is a low bar for a principle to clear, it is still nontrivial: not all subsystems of second order arithmetic express closure conditions. If every Π^1_2 sentence expresses a closure condition, then this does at least give us a weak but nontrivial syntactic criterion for falling under this concept. Moreover there is a clear sense in which all closure conditions are the same kind of thing: if weak König's lemma and Ramsey's theorem for pairs are not in the same class of principles, they do at least bear a family resemblance to one another. With this in mind I contend that not only is weakly satisfying the unity condition sufficient to make the view that reversals track closure conditions a viable account, but that weak satisfaction of this condition is all that one can expect of an account of set existence principles that accommodates not only weak König's lemma and its weakenings, but the rest of the Zoo as well.

As we have seen, there is an inherent tension between comprehensiveness on the one hand, and strong unity and nontriviality on the other. This should lead us to conclude that if we are to accommodate the data then we are unlikely to end up with an account that can strongly satisfy the unity condition. Moreover, strong unity simply doesn't seem to be a feature of the class of mathematically natural systems. The different principles, although they have features in common, have different properties and appear to be justified in different ways.[18]

Given this tension between our desiderata for a good theory of set existence principles, and our overriding concern to explain the significance of reversals, we must balance our different concerns. I contend that given the choice, we should prefer a more general account that only weakly satisfies the unity condition but can accommodate more mathematically natural systems.

The feature of this view which allows it to both satisfy the unity condition and accommodate the counterexamples to the SEC view and its variants, namely its generality, becomes a weakness when we consider the nontriviality condition (1). The account does satisfy the letter of the law, as arithmetical statements cannot be considered to express closure conditions, and thus according to the account they do not express set existence principles. Neither do Π^1_1 statements, such as those expressing that a given recursive ordinal α

[18]This may be one lesson to be drawn from the connection between the Big Five and different foundational programmes studied in chapter 4.

is wellordered. This points to a class of possible counterexamples that could undermine the account's claim to satisfy the comprehensiveness condition (2), a problem that is addressed in §2.7. Nevertheless, it is hard to escape from the conclusion that at least every Π^1_2 statement should be considered a closure condition. After all, it is the very form of these statements—which assert that for every set $X \subseteq \mathbb{N}$ of a certain sort, there exists a set $Y \subseteq \mathbb{N}$ of a different sort—that brought us to consider the view that reversals track closure conditions in the first place.

In fact, the situation is more serious than it initially appears. Thus far we have only considered closure conditions with Π^1_2 formulations, but not even all of the Big Five have Π^1_2 definitions. In particular, Π^1_1-CA_0 is not equivalent over ATR_0 to any Π^1_2 statement, although it is straightforwardly expressed as a Π^1_3 sentence [Marcone 1996, corollary 1.10]. There are even theorems of topology which exceed the strength of Π^1_1 comprehension, such as "every countably based MF space which is regular is homeomorphic to a complete separable metric space" which is equivalent to Π^1_2 comprehension [Mummert and Simpson 2005]. Such theorems will not be expressible as Π^1_3 statements, so we must consider yet more complex sentences as also expressing closure conditions if we are to bring them into the account. Since Π^1_{n+2} statements express closure conditions for Π^1_n relations, extending the account that reversals track natural closure conditions to include all Π^1_{n+2} statements seems like an obvious and well-motivated step. But this makes the account's apparent violation of the spirit of the nontriviality condition (1) even more acute, since it is not merely all Π^1_2 statements we have to worry about, but Π^1_n statements for all $n \geq 2$. In the next section we shall consider some restrictions on the class of closure conditions which might allow the account to avoid the charge of triviality.

2.6 Naturalness

We have already seen one distinguished class of subsystems of second order arithmetic, namely the *mathematically natural* ones which are equivalent to one or more core mathematical theorems. Unfortunately, this notion will not help us to resolve the weakness of the view that reversals track closure conditions with respect to the nontriviality condition (1). In particular, mathematical naturalness cannot serve as an explanation of the Big Five phenomenon: it give us no insight into why these systems, and not others, are the mathematically natural ones. What is needed instead is some notion of naturalness that serves to thin out the class of admissible closure conditions, such that we could then show that this new class of systems contains or equals the mathematically natural ones.

As we have seen in the cases of SEC and SECS, highly restrictive accounts of what set existence principles are appear highly vulnerable to counterexamples. One response is to appeal to the much used but rarely explained distinction between natural formal theories and artificial ones created by applications of diagonalisation. To avoid confusion with the related but distinct concept of mathematical naturalness defined in section 2.2, let us call this *logical* or *combinatorial naturalness*, since the axiom systems in question typically have a combinatorial or computability-theoretic flavour. The revised version of the view then holds that the significance of reversals lies in their tracking logically natural closure conditions on $\mathcal{P}(\mathbb{N})$.

The property of mathematical naturalness is one that is at least prima facie dependent on a parameter, namely the base theory. Logical naturalness is more freestanding, since it is an intensional notion: grasping that a theory is logically natural simply requires grasping the concepts involved in its statement, while mathematical naturalness is given extensionally, in terms of the existence of an equivalence with a theorem of core mathematics.

Logical naturalness is a very broad notion. It does answer the triviality concern, but only just. Apart from concerns that one might have over the very coherency of the concept, it is unclear whether adopting it does much to assuage the concerns voiced in the previous section that the view that reversals track closure conditions provides a satisfactory answer to the question of the significance of reversals.

2.7 Exceptional principles

The view that reversals track closure conditions on $\mathcal{P}(\mathbb{N})$ is a more satisfactory one than the SEC interpretation of the standard view, since it can accommodate Π^1_2 counterexamples such as weak König's lemma. All of the Big Five are naturally understood as closure conditions, with some caveats in the case of Π^1_1-CA_0, namely that it cannot be expressed as a Π^1_2 sentence but only as a Π^1_3 assertion; for details see corollary 1.10 of Marcone [1996]. Moreover, it is to be hoped that this view is indeed non-trivial in the sense of the preceding section. However, there is an additional class of exceptional principles which do not express closure conditions, namely Π^1_1 assertions expressing that some recursive linear order is a wellordering. A typical example is the statement $\mathsf{WO}(\omega^\omega)$, which asserts that the recursive set W coding a linear order $<_W$

isomorphic to ω^ω, in fact codes a wellordering. Since ω^ω is the proof-theoretic ordinal of RCA_0, this statement cannot be proved in RCA_0, or indeed in WKL_0.

That there are statements which neither express closure conditions nor are provable in the base theory RCA_0 is not, in and of itself, problematic for the account. What does cause difficulty is the fact that there are theorems of ordinary mathematics which are equivalent over RCA_0 to some of these statements—in other words, there are mathematically natural systems of this sort. The most striking example is the Hilbert basis theorem, which Simpson [1988b] showed to be equivalent over RCA_0 to $\mathrm{WO}(\omega^\omega)$.

The Hilbert basis theorem is a fundamental result in algebra. Its non-constructive character was thought remarkable at the time of its discovery, although its role in mathematical history is somewhat over-mythologised, as one can see by consulting McLarty [2012]. The equivalence between the Hilbert basis theorem and $\mathrm{WO}(\omega^\omega)$ should therefore be taken very seriously as a potential counterexample to the view that reversals track closure conditions.

In the same paper Simpson also shows that the Robson basis theorem, a generalisation of the Hilbert basis theorem, is equivalent over RCA_0 to the statement that ω^{ω^ω} is wellordered. Another mathematically natural statement of this sort can be found at the level of the small Veblen ordinal $\vartheta\Omega^\omega$. Rathjen and Weiermann [1993] showed that the graph-theoretic result known as Kruskal's Theorem is equivalent to the statement that this ordinal is wellfounded. The resulting theory $\mathsf{WO}(\vartheta\Omega^\omega)$ is strictly intermediate between ATR_0 and $\Pi^1_1\text{-}\mathsf{CA}_0$ in terms of consistency strength, and is incomparable with ACA_0 and ATR_0 in terms of proof-theoretic strength.

All of this evidence points quite clearly to the conclusion that there is a hierarchy of mathematically natural Π^1_1 statements which assert that certain ordinal notations do in fact characterise wellorderings. This presents a serious problem for the view that reversals track closure conditions since such statements are transparently not closure conditions on $\mathcal{P}(\mathbb{N})$. A wellordering statement of this sort is a universal statement asserting that, given some fixed set W that codes a recursive linear order $<_W$, no set X codes an infinite descending sequence in $<_W$. In other words, it says that a certain class of sets does not exist, and it is difficult to see how such a statement could be considered a closure condition.

Systems such as $\mathsf{WO}(\omega^\omega)$ are, however, also problems for other versions of the standard view. Since they are neither comprehension schemes nor separation schemes, wellordering statements are counterexamples to SEC and SECS just as surely as they are counterexamples to the view that reversals track

$$
\begin{tikzcd}
 & \Pi^1_1\text{-}\mathsf{CA}_0 \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\
\mathsf{WO}(\vartheta\Omega^\omega) \arrow[r, dotted] \arrow[d, Rightarrow] & \mathsf{ATR}_0 \arrow[d, Rightarrow] \\
\mathsf{WO}(\varepsilon_0) \arrow[r, dotted] \arrow[d, Rightarrow] & \mathsf{ACA}_0 \arrow[dl, Rightarrow] \arrow[dd, Rightarrow] \\
\mathsf{WO}(\omega^{\omega^\omega}) \arrow[d, Rightarrow] & \\
\mathsf{WO}(\omega^\omega) \arrow[r, dotted] \arrow[dr, Rightarrow] & \mathsf{WKL}_0 \arrow[d, Rightarrow] \\
 & \mathsf{RCA}_0
\end{tikzcd}
$$

Figure 2.2: A comparison between the Big Five and the wellordering principles discussed in this section. $A \Rightarrow B$ means that A proves the axioms of B but not conversely, while $C \dashrightarrow D$ means that C proves the consistency of D.

closure conditions. Moreover even a naïve view that leaves the notion of set existence principle largely unanalysed is likely to find such examples problematic, since they simply do not look like set existence principles—if anything, they seem to be set non-existence principles.

At this point there seem to be two strategies available for the partisan of the standard view. The first is to find a way to rule out these Π^1_1 counterexamples—that is, to find some reason to consider them not as theorems of ordinary mathematics or otherwise outside the scope of reverse mathematical analysis. The second is to retreat to a more modest thesis about what the standard view—that reversals track set existence principle—is intended to accomplish. I shall deal with both views in turn.

While $\mathsf{WO}(\omega^\omega)$ is not provable in RCA_0, it is provable in $\mathsf{RCA}_0 + \Sigma^0_2$-$\mathsf{IND}$. Similarly, $\mathsf{WO}(\omega^{\omega^\omega})$ is provable in $\mathsf{RCA}_0 + \Sigma^0_3$-$\mathsf{IND}$. Sometimes strengthened induction principles are even needed in order to prove results that look like more straightforwardly reverse mathematical results. Neeman [2011] demonstrates a case where Σ^1_1 induction is needed in order to show that Jullien's indecomposability theorem implies the weak Σ^1_1 choice scheme.

One response to these counterexamples might be to increase the strength of the induction principle used in the base theory. This is not unprecedented: in Friedman's first paper on reverse mathematics, the systems studied included the full induction scheme [Friedman 1975]. Even so, this manoeuvre appears highly ad hoc, since we seem to be stipulating that the amount of induction present in the base theory can be increased arbitrarily in order to wipe out the counterexamples to the standard view. In doing so we are failing to accommodate the data, namely that important theorems of ordinary mathematics are equivalent to such wellordering statements. We have no principled reason to pretend that they do not exist—on the contrary, a good theory of reversals should explain these equivalences, just as much as it should explain the equivalences of theorems of analysis or descriptive set theory.

Here is one attempt to justify this strategy; there may be others. The standard natural numbers ω satisfy induction for the full language of second order arithmetic, as well as for any higher-type extension. Restricting attention to ω-models can be thought of as the ideal limit of increasing the amount of induction available in the base theory. Moreover, ω-logic is complete for Π^1_1 sentences: if a wellordering statement of the type under discussion is true, then it is true in all ω-models.

Building such a presupposition into our base theory is essentially the move suggested by Shore [2010, 2013]. I discuss Shore's programme at length in chapter 5. This brute force approach simply rules out the counterexamples, but the price to be paid seems very high: again, what reason do we have to think that these equivalences are not worth considering?

These facts motivate the following view of the situation. Arithmetical statements, concerned with the properties of the natural numbers—that is, with finite objects—are decided by taking this ideal limit of induction principles, namely adopting ω-logic. Most equivalences in reverse mathematics are left untouched by this suggestion, since their primary concern is not finite objects but countably infinite ones: real numbers, countable fields, codes for Borel sets or complete separable metric spaces. The Π^1_2 theorems that form the bulk of statements studied in reverse mathematics are simply of a different kind to statements that are either explicitly finitary or are negative statements about infinite objects.

While wellordering statements like $\mathsf{WO}(\omega^\omega)$ are exceptional, that they can't be accommodated within the explanatory framework of the standard view is understandable given that they differ significantly from the other statements studied in reverse mathematics. The correct response to them is therefore modesty. Although not all mathematical theorems studied in reverse mathematics have this form, the study of $\Pi^1_{n\geq 2}$ statements concerning countable or countably representable objects forms the core of the subject, and the view that reversals track closure conditions is best understood as an attempt to explain the significance of that core.

2.8 Conclusion

There is a tension between the two primary desiderata of any account of the significance of reversals. The first is that the account should have something to say about all mathematically natural subsystems of second order arithmetic, rather than unaccountably falling silent when faced with something unexpected. This is precisely the demand placed by the comprehensiveness condition (2). The second is that it should answer the significance question by providing an explanation of what the existence of a reversal tells us. It seems reasonable to expect that any answer to the significance question will be nontrivial (1), and that it will be uniform: the explanations it provides will be unified (3).

Unfortunately, if we aim to provide an account that gives us a substantial answer to the significance question, and from which we can infer a considerable amount about the significance of reversals to particular systems, then on the evidence so far, we will find any number of counterexamples that undermine the generality of our account. This is precisely the problem faced by SEC and its variants.

If, on the other hand, we aim to provide a thoroughly general account of the significance of reversals and thus satisfy the comprehensiveness condition, we are unlikely to be able to provide a substantial or informative account of the significance of individual reversals. Walking the line between triviality and noncomprehensiveness is thus a difficult task.

The view that reversals track closure conditions attempts to strike a balance closer to triviality than noncomprehensiveness. This allows the view to accommodate the most central part of the discipline, namely the study of mathematically natural Π^1_2 theorems. Although such a general account does not, by itself, offer substantial explanations of the significance of particular reversals, it does at least offer a framework within which more fine-grained theorising can be done. The explanatory power offered by SEC can be partially assimilated by acknowledging that some closure conditions are comprehension schemes, and that comprehension schemes are a family of principles with distinctive qualities, such that their necessary use in the proof of an ordinary mathematical theorem will allow distinctive kinds of explanation. However, even this highly general account has to cope with a class of exceptional principles, namely wellordering statements equivalent to theorems whose proof relies on transfinite induction along recursive ordinals of varying heights.

The revised view therefore differs from the standard view in two essential respects. Firstly, it steps back from the claim that the significance of all the equivalences proved in reverse mathematics lies in the set existence principles thereby shown to be necessary to their proof. Secondly, it offers a specific characterisation of set existence principles, as closure conditions on the powerset of the natural numbers.

On this view, the significance of core results in reverse mathematics is that they show the crucial theorems for diverse areas of ordinary mathematics require that $\mathcal{P}(\mathbb{N})$ satisfy particular closure conditions. These closure conditions can be captured by natural axioms drawn from logic (namely from recursion theory, proof theory and model theory). In spite of the diversity of the ordinary mathematical theorems studied in reverse mathematics, the closure conditions involved are few, and have clear relationships to one another—for the most part they are linearly ordered by proof-theoretic strength.

An individual reversal demonstrates the closure condition required to support a given part of ordinary mathematics, and in some sense picks out an intrinsic feature of a theorem, namely the resources required to prove it, whether that be compactness or transfinite recursion. This feature is a proof-invariant property: every proof of the theorem in question must at some point make use of this property, although it may appear in different guises (as weak König's lemma or as the Heine/Borel theorem, for example). In the next chapter, we shall explore the extent to which these seemingly robust proof-invariant properties are dependent upon representational assumptions made in the metatheory.

3

Coding and content

This chapter draws on joint research with Sam Sanders. We are currently preparing a paper covering similar ground, but the following was written by me alone. My coauthor is therefore not responsible for any mistakes herein, and may disagree with the conclusions I reach.

3.1 Semantic aspects of reverse mathematics

The standard view in the field of reverse mathematics is that equivalences between theorems of ordinary mathematics and subsystems of second order arithmetic demonstrate the set existence principles necessary for proving those theorems. A realist reading of this view is that these equivalences demonstrate that if we accept the truth of some theorem of ordinary mathematics, we must also accept the truth of the underlying set existence principle necessary to prove it. As we saw in chapter 2, there are many details that must be supplied in order to turn the broad outlines of the standard view into a compelling metaphysical and epistemological account, but its basic scaffolding has much to recommend it.

When one turns from these epistemological and metaphysical concerns to semantic ones, this scaffolding starts to look more rickety. Much of the burden of the standard view is borne by the claim that the formalisations of ordinary mathematical theorems in subsystems of second order arithmetic are faithful, in the sense that they formally capture the mathematical content of statements of ordinary mathematics. It is this faithfulness that undergirds the significance of reversals, since it is required in order to justify claims like "Arithmetical comprehension is necessary in order to prove the Bolzano/Weierstraß theorem".

Mathematics is full of a certain kind of loose talk which can, on first hearing, be philosophically confusing. When a reverse mathematician says that Brouwer's fixed point theorem is equivalent to weak König's lemma, we should understand this claim as paraphrasing something like the following: the sentence φ in the language of second order arithmetic is a faithful formalisation of Brouwer's fixed point theorem, and φ is provably equivalent over RCA_0 to weak König's lemma. My intent in this chapter is not to pick holes in the entirely understandable use of such paraphrases in reverse mathematics, but to address the more substantive issue of whether these formalisations of ordinary mathematical theorems are indeed faithful.

Formalisations can be understood as a kind of translation, from ordinary mathematical language into the formal language in question. Some aspects of this translation appear unproblematic: logical notions such as conjunction, implication and quantification translate readily enough, just so long as they are interpreted in the same way (typically classically) in both settings. Similarly, propositions concerning natural numbers and sets of natural numbers have a direct interpretation in second order arithmetic. As we shall see, a broad range of further kinds of statements—for example concerning relations $R \subseteq \mathbb{N}^k$ for any $k \in \mathbb{N}$—can be translated very directly into this setting.

Second order arithmetic is expressively constrained, insofar as its basic vocabulary is arithmetic and it only permits quantification over natural numbers and sets of natural numbers. These restrictions mean that the language of second order arithmetic does not include quantification over, for example, finite sequences $\sigma \in \mathbb{N}^{<\mathbb{N}}$ or functions $f : \mathbb{N} \to \mathbb{N}$. Nevertheless such objects are easily and faithfully coded in second order arithmetic. For the former, we represent finite sequences of natural numbers as single natural numbers, using Gödel's β function. For the latter, an n-ary function $f : \mathbb{N}^n \to \mathbb{N}$ is represented by a set X_f of natural numbers, each of which represents a sequence $\sigma = \langle x_1, \ldots, x_n, x_{n+1} \rangle$ such that for any $x_1, \ldots, x_n$ there is only one y such that $\langle x_1, \ldots, x_n, y \rangle \in X$.

Such representations of mathematical objects can be more or less direct. Those just discussed are straightforward, at least in part because the encoding type (natural numbers, sets of natural numbers) has the same cardinality as the encoded type (finite sequences of natural numbers, functions from $\mathbb{N}^k$ to $\mathbb{N}$). Nevertheless, the faithfulness of these representations must be guaranteed if we are to take seriously the view that statements concerning these codes are faithful formalisations of the ordinary mathematical statements concerning the encoded objects.

These guarantees are given by representation theorems, which state that the relevant mathematical properties of the encoded objects are preserved by encoding them in the formal system in question. Such theorems typically have the form

$$\forall X^\tau\, [\, \varphi(X) \;\Leftrightarrow\; \exists y^\rho\, \psi(y)\,]$$

where X^τ means that the object X is of type τ, and y^ρ means that y is of type ρ. The property $\varphi(X)$ is the *represented property*, while the property $\psi(y)$ is the *representing property*. Similarly we say that X is the *represented object* and y is the *representing object* or *representation*. If X has the represented property then there is an object y that represents it, and if there is a representation y with the right property, then the represented object X has the represented property. Moreover, y usually encodes enough information about X that X can be constructed from it.

An important case is that of *continuous functions* between uncountable spaces such as the reals, which are central to analysis, but due to cardinality constraints cannot be represented directly in second order arithmetic. As we shall see shortly, their representation by codes is much less direct than the examples just mentioned.

Due to their role as one of the basic objects of analysis, continuous functions have been much studied not only in the classical setting, but also in the realms of constructive, computable and nonstandard analysis. We shall start from the classical Weierstraß ε-δ definition of continuity for real-valued functions; a textbook exposition is given by Rudin [1976, p. 85]. This definition can be straightforwardly generalised to functions between metric spaces rather than just from reals to reals, but all of the central issues already arise in this fundamental case. For simplicity of exposition we shall therefore, in the main, stick with the reals.

Given $X \subseteq \mathbb{R}$ and $p \in X$, a function $f : X \to \mathbb{R}$ is *continuous on X at p* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $x \in X$,

$$|x - p| < \delta \;\Rightarrow\; |f(x) - f(p)| < \varepsilon.$$

If f is continuous at every $y \in X$, f is *continuous on X*.

Since it lacks third-order quantifiers, continuous real-valued functions cannot be represented directly in second order arithmetic as functions on the reals. The representation of real-valued functions instead leans on the representation of the real numbers in second order arithmetic as the completion of the rational numbers $\mathbb{Q}$. Individual real numbers are represented by Cauchy sequences of rational numbers with a fixed rate of convergence: $\langle q_k | k \in \mathbb{N}\rangle$ such that for all $k, i \in \mathbb{N}$, $|q_k - q_{k+i}| \leq 2^{-k}$.

Definition 3.1.1 (continuous functions)**.** Let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces and let $\phi : \widehat{A} \to \widehat{B}$ be a continuous function between them. Then a *code for* ϕ is a set of quintuples $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ which obeys the following conditions:

1. if $((a, r)\Phi(b, s)$ and $(a, r)\Phi(b', s'))$ then $d(b, b') \leq s + s'$;
2. if $((a, r)\Phi(b, s)$ and $(a', r') < (a, r))$ then $(a', r')\Phi(b, s)$;
3. if $((a, r)\Phi(b, s)$ and $(b, s) < (b', s'))$ then $(a, r)\Phi(b', s')$

where $(p, q)\Phi(r, s)$ means $\exists n((n, p, q, r, s) \in \Phi)$; $(a, r) < (a', r')$ means $d(a, a') + r' < r$; and d is the metric on $\widehat{A}$.

So $\widehat{A}$ could be, for example, a closed interval and $\widehat{B}$ the reals with their usual metric. After presenting this definition, Simpson [2009, p. 85] gives the following gloss on it:

> Recall ... that $B(a, r)$ denotes the basic open ball centered at a with radius r. Intuitively, $(a, r)\Phi(b, s)$ is a piece of information to the effect that $\phi(x) \in$ the closure of $B(b, s)$ whenever $x \in B(a, r)$, provided $\phi(x)$ is defined.

Even with this intuitive picture in hand, this is a complex definition, and hard to parse on first reading. But by encoding continuous functions in this way one can overcome the expressive limitations of RCA_0 and formalise the central analytical notion of continuity, albeit in a highly indirect manner.

3.2 Enriched definitions and constructivity

One way to understand the hierarchy of mathematically natural systems extending RCA_0 is as standard yardsticks that allow us to measure the degree of nonconstructiveness of the theorems that are provably equivalent to them, although such an understanding does require that we interpret nonconstructiveness in a particular way, namely in terms of uncomputability: with mathematics in RCA_0 understood as corresponding to computable mathematics, the systems that extend RCA_0 correspond to principles asserting the existence of different classes of uncomputable sets. So the separable Hahn/Banach theorem is nonconstructive because it implies the existence of uncomputable sets—but the Bolzano/Weierstraß theorem is more nonconstructive, because it implies the existence of the Turing jump of every set, which is a stronger principle than weak König's lemma.

Simpson [2009, p. 32] argues that constructive mathematicians respond to such nonconstructive theorems by conceptual change: enriching definitions in such a way that the formerly classical theorems stated in terms of these notions become constructively provable.

> The typical constructivist response to a nonconstructive mathematical theorem is to modify the theorem by adding hypotheses or "extra data". In contrast, our approach in [Simpson 2009] is to analyze the provability of mathematical theorems as they stand, passing to stronger subsystems of Z_2 if necessary.

In Bishop's constructive analysis, the classical definition of a continuous function from the preceding section is supplemented by an additional hypothesis: that every such function is associated with a *modulus of uniform continuity*. A real-valued function f on a compact interval X is continuous, in Bishop's sense, if there exists a modulus of continuity ω_f such that for every $\varepsilon > 0$ the value $\omega_f(\varepsilon) > 0$, and

$$|x - y| < \omega_f(\varepsilon) \;\Rightarrow\; |f(x) - f(y)| < \varepsilon$$

for every $x, y \in X$ [Bishop and Bridges 1985, p. 38].

The existence of (codes for) such moduli of uniform continuity is not in general provable in RCA_0, although many important special cases are provable; Simpson [2009, pp. 136–7] remarks that

> it is interesting to note that "any continuous function [from $\mathbb{R}^k$ into $\mathbb{R}$] which arises in practice" can be proved in RCA_0 to have a modulus of uniform continuity on any closed bounded subset of its domain.

In other words, RCA_0 suffices in the typical cases that mathematicians are interested in. Simpson [2009, p. 137] goes on to say (original emphasis preserved) that

> This situation has prompted some authors, for example Bishop and Bridges [1985, p. 38], to build a modulus of uniform continuity into their definitions of continuous function. Such a procedure may be appropriate for Bishop since his goal is to replace ordinary mathematical theorems by their "constructive" counterparts. However ... our goal is quite different. Namely, we seek to draw out the set existence assumptions which are implicit in the ordinary mathematical theorems *as they stand*. ... Thus Bishop's procedure would not be appropriate for us.

Simpson, then, is very clear that one criterion for the faithfulness of a formalisation of an ordinary mathematical theorem is the absence of such enrichments. This seems right: if a formalisation is to capture the content of a mathematical statement then supplementing its hypotheses with "extra data" such as a modulus of uniform continuity appears to change the meaning of the statement. Nevertheless it is important to be clear on two points, the first of which is that an apparent enrichment is not always a genuine enrichment: a statement that employs an enriched notion may turn out to be equivalent to an alternative formalisation that does not.

Moreover, such semantic change when formalising ordinary mathematical statements is not confined to constructivists; reverse mathematics itself contains many examples of this phenomenon, simply because, as in the constructivist case, the theory in which one typically works (RCA_0) is proof-theoretically and expressively weak. Consider the notion of a *structure* in model theory. This is usually understood as a set (the domain), together with a collection of constants drawn from the domain, and a collection of functions and relations on the domain. For each such structure M there exists a uniquely defined satisfaction relation $M \models \varphi$, which is defined for all formulas φ in the language of M. Second order arithmetic can only handle countable structures, but the key metatheoretic results for first-order logic show that this is not a serious restriction, as any consistent theory in a countable language has a countable model. However, if we formalise the notion of a countable structure in a direct way that closely matches the usual model-theoretic definition, then the base theory RCA_0 is too weak to prove most model-theoretic results, because it cannot prove that for each countable structure M, the satisfaction relation for M exists. To prove this statement we actually need a system known as ACA_0^+, which extends the axioms of ACA_0 with the principle that the Turing jump operator can be iterated along ω.

Worse still, many model-theoretic statements, including such central results as the compactness and completeness theorems, turn out to be weaker than ACA_0. In order to do reverse mathematics one needs to work over a base theory that cannot prove the theorems whose strength is being proved, such as RCA_0, so one needs to replace the standard definition of a countable structure with one that can be better handled in the base theory. As it turns out, the way to do this is to build the entire elementary diagram—the set of first-order sentences true in that structure—into the definition of a countable structure (for details see §II.8 of Simpson [2009]). It is using this enriched definition of a countable structure that results such as the completeness theorem are proved to be equivalent to WKL_0 over RCA_0. We may reasonably ask whether the elementary diagram of a structure is more essential to a structure than the modulus of uniform continuity is to a continuous function.

3.3 Higher order reverse mathematics

In order to make these concerns precise, and gauge the strength of the representation theorems necessary to vindicate the coding choices made in reverse mathematics, Ulrich Kohlenbach introduced higher types to reverse mathematics [Kohlenbach 2002, 2005]. Kohlenbach's system includes all finite types, and thus allows statements about higher-type objects—such as functions on the reals—to be formalised directly. This makes it possible to compare different representational approaches, and understand the higher-order commitments implicit in the use of reverse mathematical coding devices.

In order to do this, we must briefly outline the essentials of Kohlenbach's system, which is described in full in §2 of Kohlenbach [2002], with a briefer but more accessible presentation in §2 of Kohlenbach [2005]. The set $\mathbf{T}$ of finite types contains a type 0, and for every pair of types ρ and τ, it also contains the type $\rho \to \tau$ of functions from ρ to τ. The type 0 is the type of the natural numbers $\mathbb{N}$, while the type 1 is the type of functions $f : \mathbb{N} \to \mathbb{N}$, so it roughly corresponds to the sets of natural numbers in the usual formulation of second order arithmetic.[19] The underlying logic for theories in this language is classical, many-sorted logic. Where necessary in the rest of this chapter, a variable's type will be made clear by a superscript, so "x^τ" denotes a variable of type τ.

With different types in play, it is often crucial to be able to form sets of elements of different types. To this end a family of choice schemas can be formulated in higher order mathematics. The *schema of quantifier-free choice for the types* ρ, τ is given by

$$\text{QF-AC}^{\rho,\tau} \;\equiv\; (\forall x^\rho \exists y^\tau \varphi(x,y)) \to (\exists F^{\rho\to\tau} \forall x^\rho \varphi(x, Fx))$$

and the full *schema of quantifier-free choice* for all types is given by

$$\text{QF-AC} \;\equiv\; \bigcup_{\rho,\tau\in\mathbf{T}} \text{QF-AC}^{\rho,\tau}$$

where $\mathbf{T}$ is the set of all finite types.

[19] A slightly closer analogue is the second order functional calculus used in Grzegorczyk et al. [1958] and other papers from that period.

One of the fundamental theories formulated in Kohlenbach's system is known as E-PRA$^\omega$, and it is effectively an analogue of primitive recursive arithmetic PRA, but expressed in the language of all finite types. By extending E-PRA$^\omega$ with the axiom of quantifier free choice for the type 0, QF-AC^{0,0}, one obtains the system RCA_0^ω. This system proves Σ^0_1 induction and Δ^0_1 comprehension, the defining axioms of RCA_0. Since it is formulated in a language with functions rather than sets, RCA_0^ω is not strictly speaking an extension of RCA_0. However, by identifying sets with their characteristic functions one can interpret RCA_0 as a subsystem of RCA_0^2, the second order fragment of RCA_0^ω.

RCA_0^ω is conservative over RCA_0^2 for sentences in the second order fragment of the language, and thus in an obvious sense also over RCA_0 [Kohlenbach 2005, proposition 3.1]. As a corollary of this result, the usual hierarchy of subsystems of second order arithmetic can all be formulated in this setting, and the usual relationships between them hold. RCA_0^ω can also be extended with the axiom schema of full induction, yielding the stronger theory RCA^ω. This is, in effect, a higher-type version of RCA, i.e. RCA_0 plus the full induction scheme (1.13).

Before we move on to the substantial results obtained within this framework, let us briefly pause to consider two set existence axioms related to subsystems of second order arithmetic. The first is $(\mathcal{E}_1)$, which asserts the existence of a functional E_1 that allows one to determine whether or not two reals $x, y \in \mathbb{N}^\mathbb{N}$ are equal.[20] The system $\mathsf{RCA}_0^\omega + (\mathcal{E}_1)$ implies and is conservative over ACA_0 [Hunter 2008, theorem 2.5]. The second is $(\mathcal{E}_2)$, which is just $(\mathcal{E}_1)$ but for functions on the reals. The system $\mathsf{RCA}_0^\omega + (\mathcal{E}_2)$ implies and is conservative over Π^1_∞-CA_0 [Hunter 2008, corollary 2.6], i.e. full second order arithmetic Z_2.

3.4 The strength of representations

The formal counterparts of ordinary mathematical theorems concerning continuous functions generally turn out, when formalised in second order arithmetic, to be either provable in RCA_0 (such as the intermediate value theorem), or equivalent to one of ACA_0 (the Ascoli lemma) or WKL_0 (Brouwer's fixed point theorem). Since continuous real-valued functions cannot be directly represented within second-order arithmetic, these results rely on the representation of such functions by codes. Following Kohlenbach [2002] and Sanders [2015] we refer to these codes as *RM-codes*.

In Kohlenbach's higher order reverse mathematics, continuous functions are directly representable as type-2 functionals. For functionals $\Phi : X \to Y$ where

[20] Kohlenbach [2005] calls this $(\exists^2)$.

X and Y are Polish spaces (complete separable metric spaces), Φ is continuous in the usual ε-δ sense just in case it is sequentially continuous. This result is provable in RCA_0^ω plus a stronger choice principle, QF-AC^{0,1}, which is needed to prove the implication from sequential continuity to ε-δ continuity [Kohlenbach 2002, proposition 4.1 and remark 4.2].

Continuity in the reverse mathematics sense is another matter entirely. Given an RM-code g, RCA_0^ω + QF-AC^{0,1} proves that the direct representation of the continuous function coded by g exists. The converse—that every directly represented continuous function has an RM-code—is not provable in RCA_0^ω+QF-AC^{0,1}. Adding the full induction schema and the full quantifier-free choice schema does not help: E-PA$^\omega$+QF-AC does not prove the representation theorem either. This already appears problematic, since one of the apparent epistemic advantages of using a weak base theory like RCA_0 is that one can demonstrate from a vantage point with limited theoretical commitments that a given axiom is necessary to the proof of some theorem. But from Kohlenbach's results we can see that the theoretical resources available to RCA_0, even when its expressive resources are enhanced in order to directly represent the higher-order objects that can only be indirectly coded in second order arithmetic, are not sufficient to prove the faithfulness of the representation in question, that is, that every continuous function has an RM-code. Moreover, as the following theorem demonstrates, the representational assumption implicit in reverse mathematical practice yields an enrichment of the direct representation by a modulus of pointwise continuity.

Theorem 3.4.1 (Kohlenbach)**.** *Let Φ^2 be a continuous functional from the Baire space to $\mathbb{N}$ (both with the usual metrics). Then the following are pairwise equivalent over RCA_0^ω:*

1. *There exists an RM-code of Φ;*
2. *There exists a continuous modulus of pointwise continuity for Φ.*

The upshot is that the reverse mathematics definition in RCA_0 of continuous functions implies, in the higher order setting, a constructive enrichment of the direct representation of continuous functions in the sense of section 3.2. Sanders [2015] extends Kohlenbach's work to show that the reverse mathematics definition of continuity gives rise to a nonstandard enrichment of continuity, and that the nonstandard continuous type-2 functionals are precisely those with RM-codes.

In the above theorem the range of the continuous function was the natural numbers, but more typically we are interested in functions on spaces like the

Baire space, the Cantor space, the reals and the like. Using a surprising decidability result of Dag Normann, Kohlenbach proved that if we restrict ourselves to a particular class of spaces—namely the Cantor space and, more generally, compact Polish spaces—then the standard reverse mathematical coding of continuous functions between these spaces is faithful.

More precisely, WKL_0^ω proves that the restriction of every (direct representation of a) continuous functional $\Phi^{1\to 1}$ to the Cantor space (or any compact Polish space) has an RM-code [Kohlenbach 2002, proposition 4.10]. This appears to salvage the reverse mathematics of continuous functions for WKL_0: since the higher-order variant of WKL_0 proves that for compact Polish spaces, continuous functions have RM-codes, this representation theorem becomes just that—a theorem—when working with WKL_0. We therefore end up with a rather mixed report on the faithfulness or otherwise of the reverse mathematical representation of continuous functions. When working in RCA_0^ω the representation is indeed enriched, but for important spaces working in WKL_0 is sufficient to guarantee the existence of RM-codes for continuous functions.

There are some important open questions still to be answered. The first concerns the provable existence of RM-codes. We know that arithmetical comprehension in the higher-type setting suffices to prove that every continuous function between Polish spaces has an RM-code: does WKL^ω, i.e. E-PA$^\omega$ + WKL, suffice to prove this result? If it does, this would seem to show that the representation of every continuous function between Polish spaces by RM-codes is faithful, relative to WKL. Secondly, can we weaken the theory needed to prove the existence of RM-codes for continuous functions on the Cantor space from WKL_0^ω? Here the obvious target is WWKL_0, which is a mathematically natural system intermediate in strength between RCA_0 and WKL_0. Amongst other statements involving continuity, WWKL_0 is equivalent to the statement that every continuous bounded function is Riemann integrable (where "continuous" is understood in the usual reverse mathematics sense).

If one thing is clear from this investigation, it is that the faithfulness of a representation is relative to the principles one accepts in one's metatheory. For the reverse mathematics of statements involving continuous functions, the reverse mathematician appears committed to at least having weak König's lemma available in her metatheory, and perhaps even a stronger theory that validates the existence of RM-codes for all continuous functions between complete separable metric (Polish) spaces.

We now turn to a different case study, namely the reverse mathematics of general topology, where the subtle representational problems discussed above become glaring. Despite its importance in mathematics since its inception in the early 20th century, general topology must count as one of the fruits of the set-theoretic revolution, and thus seems to lie outside "ordinary mathematics" as Simpson [2009] conceives of it. Its set-theoretic roots certainly engender difficulty in the usual reverse mathematics setting, because given a set of points X, a topological space on X is a higher-type object: a set of subsets of X. Typically one wants to study topologies at least on the Baire space ω^ω, if not higher spaces as well, but to study topological spaces in second order arithmetic they must be countably representable.

Mummert and Simpson [2005], in a paper initiating the reverse mathematics of general topology, study a certain kind of topological space (MF spaces) with a *countable basis* that generates the topology, and thus containing all the information necessary to represent the space. *Separable spaces*—those with a countable subset that is dense in the space—are another example of topological spaces that are countably representable. It is therefore at least possible to code certain topological spaces in second order arithmetic, although the base theory typically needs to be strengthened beyond RCA_0.

From the perspective afforded us by Kohlenbach's work it is natural to ask how strong the representational assumptions that underpin this use of countable bases are. This question has been answered by James Hunter, who shows that they are very strong indeed [Hunter 2008, proposition 2.15].

Theorem 3.4.2 (Hunter)**.** *The existence of a type-3 set of type-2 objects with cardinality $\leq \beth_1$ is equivalent to $(\mathcal{E}_2)$.*

The axiom $(\mathcal{E}_2)$, first mentioned at the end of the preceding section, is extremely strong: it implies the second order comprehension scheme Π^1_∞-CA. In other words, the higher order framework reveals that the existence of a countable representation of a topological space with cardinality $2^{\aleph}_0$ or greater is sufficient to imply full second order arithmetic Z_2. As Hunter also proves, the statement that a separable topological space exists is also equivalent to $(\mathcal{E}_2)$, this time over the base theory $\mathsf{RCA}^\omega_0 + (\mathcal{E}_1)$.

This means that, save in one crucial respect, the situation for topology mirrors that for continuous functions. In both cases a representational assumption is made in the metatheory, and in both cases that assumption outstrips the strength of the higher type counterpart of the object theory, namely the base theory RCA_0. However, in the case of continuous functions the representational assumption does not seem too problematic, except possibly for the reverse mathematics of systems weaker than WKL_0. This is not the case for topology. A framework in which to study the reverse mathematics of general topology should allow us to address uncountable spaces, including countably representable ones. But Hunter's work shows that simply assuming the existence of countable codes for higher-order topological spaces implies full second order comprehension Π^1_∞-CA.

This dwarfs the strength of systems typically studied in reverse mathematics. For example, the main result of Mummert and Simpson [2005] is that the statement "Every countably based regular MF space is homeomorphic to a complete separable metric space" is equivalent to Π^1_2-CA_0. This was first time that a theorem of "core mathematics" had been shown to be equivalent to Π^1_2-CA_0, and it was striking because the proof-theoretic strength of this theorem is substantially greater than the bulk of results in reverse mathematics.[21] Even so, it is still far weaker than full second order arithmetic Z_2.

Employing a representational assumption that is much stronger than the theorems that actually use that representation seems prima facie problematic. Articulating precisely why it is problematic is more complex. We should first note the platonistic attitude running through reverse mathematical practice: when working in a weak theory, stronger principles are typically assumed to be true and thus available as an extension where necessary. A reversal, for example, might be true but unprovable in RCA_0—at which point it is entirely legitimate to strengthen one's base theory in order to prove it. Reverse mathematics is an exercise in "How little can we get away with?", but in the metatheory anything goes, and the full range of set-theoretic truths and techniques are available.

In particular, the representation theorems that allow one to work with countable objects in place of uncountable metric spaces and the like are true in this set-theoretic backdrop. As a result, whilst being aware that coding introduces some subtle issues, reverse mathematicians are quite comfortable using these intricate representational devices. This attitude is problematic insofar as we take reverse mathematical results to be demonstrating something profound about theorems of ordinary mathematics, namely the principles required to prove them. A provable equivalence (over a weak base theory) between an axiom system S and a formalisation τ of an ordinary mathematical theorem T is significant because it shows that the axioms of S are necessary in order to

[21] Mummert and Simpson [2005] do not clarify why they use the term "core mathematics" rather than "ordinary mathematics", but as Simpson [2009] explicitly excludes general topology from ordinary, non-set-theoretic mathematics, it could be that they wish to suggest that while it may not be an ordinary mathematical statement, the theorem they study is from mathematics proper rather than having a metamathematical or explicitly set-theoretic character.

prove T. This is not supposed to be merely a fact about the formal statement τ, but about the ordinary mathematical theorem T.

If the resources involved in the formalisation of T as τ are greater than those of (an appropriate conservative extension of) the base theory, then it would seem that the base theory is not adequate to express T, since it cannot prove the faithfulness of the notions involved. This requirement is quite strict: the standard reverse mathematics representation of continuous functions would not meet it, since RCA_0^ω does not prove the existence of an RM-code for every continuous function. We can draw a less strict variation from Kohlenbach's argument that since the representation theorem for continuous functions is provable in WKL_0^ω, the representation is adequate for the reverse mathematics of WKL_0 (but possibly not WWKL_0). The variation is as follows: given a theorem T of ordinary mathematics, formalised as a statement τ in the language of second order arithmetic, the faithfulness of every mathematical notion contained in T must be provable in an appropriate conservative extension of the system $\mathsf{RCA}_0 + \tau$ (or more generally, for a base theory B, $B + \tau$).

In most cases where such a τ is not provable in the base theory, it has turned out to be equivalent to one of the Big Five extending RCA_0, so the relevant representation theorems would in practice usually have to be proved in the higher type version of one of WKL_0, ACA_0, ATR_0, or Π^1_1-CA_0. While this allows us to salvage the reverse mathematics of continuous functions for systems $T \supseteq \mathsf{WKL}_0$, it does have a nasty side-effect, namely inducing a further relativity in the justifiability of representation theorems. We can no longer take the faithfulness of the representation in RCA_0 of some higher-type mathematical notion to be guaranteed by the same principles that are accepted in the base theory, and thus proceed to determining the proof-theoretic strength of theorems involving that notion without further inquiries as to the status of the relevant representation theorem. Instead, we must determine that the representation theorem is sanctioned by some appropriate conservative extension of each system proved equivalent to a theorem involving that notion. This leaves some existing results, such as equivalences between statements about continuous functions and WWKL_0, in rather murky water.

One way to read these results is as a vindication of Feferman's explicit mathematics [Feferman 1975a, 1977]: we should formalise ordinary mathematics in expressively adequate theories of higher types, and then reduce these theories to more basic ones, in the spirit of the relativised Hilbert programme [Feferman 1988]. Since we must directly formalise higher-type objects anyway, in order to determine that we are not smuggling strong axioms in through the back door, we might do better to go via Feferman's route, rather than trying to squeeze higher-type objects into second order arithmetic even when, like topological spaces, they clearly don't fit.

We close with an aside. Downey, Hirschfeldt, Lempp, and Solomon [2002] study the reverse mathematics of the Nielsen–Schreier theorem that every subgroup of a free group is free. They show that if one formalises subgroups as sets, then the Nielsen–Schreier theorem is provable in RCA_0, but if one formalises subgroups as being given by generators, then it is equivalent to ACA_0. Here, the representations are quite direct, so the issue is not whether the representations themselves are problematic in the same sense as those for continuous functions or topological spaces, but rather what the correct formalisation of the concept of a subgroup is—or, indeed, whether there is such a thing.

4

Foundational analysis

4.1 Reverse mathematics and foundations

The main philosophical role attributed to reverse mathematics in the current literature is what I shall call *foundational analysis*. This application has been strongly promoted by Stephen Simpson, born out of his view (stated amongst other places in his [2009] and [2010]) that there is a correspondence between subsystems of second order arithmetic and foundational programmes such as Weyl's predicativism and Hilbert's finitistic reductionism. By providing a hierarchy of comparable systems, and proving the equivalence of theorems of ordinary mathematics to these systems, reverse mathematics demonstrates what resources a particular theorem requires, and what a given system cannot prove. In other words, when committing to a foundational system reverse mathematics lets us know precisely what we are giving up. It also tells us when a proponent of such a system employs mathematical resources that she is not entitled to, as they go beyond what her preferred foundation can prove. By applying reverse mathematics to questions of this sort we can determine the degree to which ordinary mathematics can be recovered by proponents of these foundational theories, hence my use of the term foundational analysis.

The following example should clarify the notion of foundational analysis. Suppose Sarah is a predicativist in the tradition of Weyl. She believes that the natural numbers form a completed, infinite totality, and that sets which can be defined arithmetically—i.e. with quantifiers ranging over the natural numbers, but not over sets of them—also exist. This would lead her to accept the arithmetical comprehension scheme, and thus the subsystem of second order arithmetic ACA_0. She might even accept a somewhat stronger system; this possibility is explored in §4.4. But given Sarah's predicativist outlook she would resist the thoroughly impredicative axiom scheme of Π^1_1 comprehension, and its associated subsystem of second order arithmetic $\Pi^1_1\text{-}\mathsf{CA}_0$.

Now suppose that her colleague Rebecca disagrees with Sarah's predicativism and wants to persuade her that it is an inappropriate foundation for mathematics. She might argue as follows: While Sarah accepts ACA_0 and perhaps even some stronger subsystems of second order arithmetic, she will not accept Π^1_1-CA_0. On the other hand, since Sarah wants her predicativist outlook to provide a foundation for all of mathematics, it would be strange if she failed to account for important theorems of ordinary mathematics—say, in abelian group theory. Consider the statement "Every countable abelian group can be expressed as a direct sum of a divisible group and a reduced group". The group theorist in the street, Rebecca argues, believes this to be true. Sarah might tentatively agree, whereupon Rebecca would point out the following theorem from reverse mathematics: assuming that every countable abelian group is a direct sum of a divisible group and a reduced group, one can prove (in RCA_0, which Sarah clearly accepts) the Π^1_1 comprehension scheme.

It appears that Sarah has some explaining to do. Either she must abandon her predicativism, or she must push back against the naturalistic line Rebecca is urging upon her. Neither course appears terribly palatable, while the fact that this theorem is drawn not from set theory or some other area of mathematics whose ontological commitments might be thought extravagant could be taken as evidence that the problem here is a pressing one. The contentious statement is an ordinary theorem from a core area of mathematics, which reverse mathematical analysis shows us to have substantial proof-theoretic strength.

Foundational analysis does not offer a knockdown argument against predicativism, or indeed any foundational view with limited theoretical resources. Rather, it makes arguments like the dispute between Rebecca and Sarah not just possible but precise: we can see, within a common framework (namely the base theory RCA_0, and the coding required to represent ordinary mathematical concepts in it), just where the boundaries of these foundational systems lie. As a rational agent, Sarah surely formed her foundational views in the full understanding that they require her to give up on any mathematics that view deems to be without foundation. The decision to give up on or stick with her foundation is not one to be taken lightly, and it is one that should be made by considering the relevant facts. These facts can, in large part, be supplied by foundational analysis, which allows Sarah and the rest of us to see precisely what is at stake.

For foundational analysis to play a useful philosophical role in mediating between disputants with different foundational stances, it must be possible to carry out this analysis on ground which is common between the disputants.

Such common ground has several aspects; amongst them we can distinguish commonality of language; commonality of premises; and commonality of rules of inference. So while a predicativist and a platonist like Sarah and Rebecca might disagree about whether Π^1_1 comprehension is a valid axiom, they both accept the laws of classical logic and at least the axioms of RCA_0, and thus both will agree that the theorem above is not predicatively provable. In other words, foundational analysis makes it clear where the fault lines lie, and the presence of common ground makes the conclusion available not just to those who accept stronger axioms or rules of inference, but those who are committed to a more limited foundational framework and will only accept mathematical conclusions derived from that framework.

Notice that Sarah already accepted that Π^1_1 comprehension was not a predicative principle, otherwise she would not have been able to deduce that the theorem about abelian groups was not predicatively provable. In accepting this Sarah goes beyond what her foundation can formally prove. If she accepts ACA_0 and no more, then she is not in a position to separate Π^1_1-CA_0 from ACA_0. This is due to Gödel's second incompleteness theorem: since

$$\mathrm{Con}(\mathsf{ACA}_0) \Rightarrow \mathsf{ACA}_0 \nvdash \mathrm{Con}(\mathsf{ACA}_0)$$

is provable in a weak system (RCA_0 is more than sufficient), as is

$$\Pi^1_1\text{-}\mathsf{CA}_0 \vdash \mathrm{Con}(\mathsf{ACA}_0),$$

we have that

$$\mathrm{Con}(\mathsf{ACA}_0) \Rightarrow \mathsf{ACA}_0 \subsetneq \Pi^1_1\text{-}\mathsf{CA}_0,$$

i.e. Π^1_1-CA_0 is a proper extension of ACA_0. We cannot eliminate the assumption of the consistency of ACA_0, since if ACA_0 is inconsistent then it proves everything that Π^1_1-CA_0 does, which is to say every sentence in the language of second order arithmetic.

The upshot of this is the fact that Π^1_1 comprehension is not a predicative principle cannot be grasped on the basis of her acceptance of Sarah's predicative formal theory, no matter how strong it is, since we can re-run the above argument for any system S such that $\mathsf{ACA}_0 \subseteq S \subsetneq \Pi^1_1$-$\mathsf{CA}_0$. For Sarah or any predicativist, the impredicativity of Π^1_1 comprehension must therefore be justified by some other means. One candidate justification might be Sarah's acceptance of the soundness of the predicative formal theory ACA_0, or a predicative extension thereof. This is an informal, metatheoretic premise along the lines of: The axioms of S are true, and the rules of inference of classical logic preserve truth, so all the consequences of S are also true, and therefore S does not prove any contradiction, and is consistent.

Alternatively, the impredicativity of Π^1_1-CA_0 might itself be taken as a basic (albeit presumably defeasible) belief. That is to say, the defining axiom of Π^1_1 comprehension appears, on the face of it, to be impredicative: it employs quantification over all sets of natural numbers, and appears to do so in an essential way. In the absence of evidence to the contrary, Sarah should assume that Π^1_1-CA_0 is an impredicative axiom system, and thus unacceptable on the basis of her predicativist stance.

Before studying the connections between foundational programmes and subsystems of second order arithmetic in more detail, let us briefly consider the ramifications of the previous chapter's findings for foundational analysis. We saw that the faithfulness of formalisations of ordinary mathematical notions depends upon representation theorems proved in the metatheory, and that these theorems can in fact be quite strong (in terms of proof-theoretic strength). If one wishes to make an argument of the kind that Rebecca does, then one tacitly relies on the faithfulness of the formalisations employed in the reverse mathematics of the particular statements one deploys as examples of ordinary, true mathematical theorems that are unprovable in a given foundational framework. If the relevant representational assumptions are strong enough that they would not be provable in the foundational framework in question, this seems to throw a spanner in the works.

For example, if Rebecca wanted to invoke theorems of topology in her attempt to persuade Sarah that predicativism is inadequate to mathematical practice, and thus mathematical truth, then her argument would appear to rely on a suppressed premise, namely the faithfulness of the representation of topologies with countable bases. Since that premise entails full second order arithmetic, Sarah could reasonably respond that the so-called theorems of topology are not, from her predicative perspective, anything of the sort: they are simply formal statements in the language of second order arithmetic that are not predicatively provable. To identify them with particular theorems of topology requires that they be faithful translations, and the proof of that faithfulness requires theoretical resources that she is on principle not willing to commit to.

Foundational analysis therefore provides us with a good reason to be strict when it comes to our demand that representation theorems be provable in (an appropriate conservative extension of) the base theory: we thereby ensure that the reverse mathematical results can be read as intended, i.e. as demonstrating the mathematical resources necessary to prove a particular theorem of ordinary mathematics. This allows the kind of naturalistic argument given by Rebecca to be understood by the proponent of a given foundation within their theoretical framework. In the absence of this understanding, such an argument would appear (to Sarah, say) to be a non-sequitur.

4.2 Computable and constructive analysis

Simpson [2009, p. 31–2] writes that the reconstruction of ordinary mathematics within the formal system RCA_0 bears a resemblance to Bishop's constructive analysis [Bishop and Bridges 1985]. One point of agreement is that RCA_0 is compatible with the assertion that every total function $f : \mathbb{N} \to \mathbb{N}$ is recursive; indeed, it is true in REC, the minimum ω-model of RCA_0. The connection between Bishop-style constructivism and RCA_0 is discussed in more detail by Friedman et al. [1983], who point out another area of compatibility. RCA_0 is, by a result of Parsons [1970], Π^0_2 conservative over primitive recursive arithmetic. Consequently, any Π^0_2 theorem of RCA_0 is constructively valid, since PRA is generally accepted as a constructive system.

Thus far we have only discussed points of compatibility, but there are ways in which work in RCA_0 draws directly on constructive analysis. Similar constructions and proofs are possible, such as the proof of the Weierstraß approximation theorem in RCA_0, which mimics a typical constructive proof from Bishop and Bridges [1985]. Interestingly, constructive mathematics is also a rich source of recursive counterexamples that have inspired classical proofs of equivalences over RCA_0. Brown and Simpson [1986]'s proof that the separable Hahn/Banach theorem implies weak König's lemma (over RCA_0) is based on a recursive counterexample originating with Bishop, while Simpson [1984]'s proof that Peano's existence theorem also implies weak König's lemma over RCA_0 is based on a recursive counterexample of Aberth [1980].

Despite these similarities, the clash of logics between the two systems makes the prospect of carrying out foundational analysis for Bishop's constructive analysis in RCA_0 untenable, as the following argument should make clear. Constructivists reject the unrestricted use of the law of the excluded middle (LEM), and consequently do not consider the classical entailment relation to preserve justification. Even if some set of axioms T are deemed constructively acceptable, a theorem φ may be rejected if it is proved from those axioms by classical rather than constructive means. Results in traditional reverse mathematics, which assumes full classical logic including unrestricted use of LEM, will therefore not always be accepted as meaningful by constructivists, depending on whether the proof in question employs these techniques. One example is the intermediate value theorem, which is provable in RCA_0 but not constructively valid in Bishop's sense, as it is equivalent to the constructively invalid *lesser limited principle of omniscience* (LLPO).

The suggestion that Simpson makes in later writing such as [2010] that the foundational programme best identified with work in RCA_0 is in fact *computable analysis* seems closer to the mark. Computable analysis is a fusion of computability theory, scientific computing, and real analysis, which aims at providing a rigorous foundation for computing solutions to mathematical questions in scientific fields such as physics where phenomena are modelled in terms of continuous functions. In computable analysis, "an algorithm is required for any entity employed [and thus] definition always goes with evaluation" [Aberth 1980, pp. 1–2]. The standard template for developing computable analysis runs as follows: one selects a model of computation on the natural numbers, and based on this choice, one develops a notion of computation on the reals, allowing one to create a framework for solving problems in computable analysis.

The approach of Aberth [1980] and Pour-El and Richards [1988] is based on classical recursion theory, and they permit unrestricted use of classical reasoning such as LEM. In this way their approach is similar to the development of classical analysis in RCA_0, and their underlying motivations are similar.[22] Pour-El and Richards [1988, p. 4] write that

> Our objective is to delineate the class of computable processes within the larger class of all processes. In this, our viewpoint is analogous to that of the complex analyst, who regards the analytic functions as a special case of the class of all functions, but regards all functions as existing mathematical objects.

Computable analysis thus differs from real analysis in that its subject matter is restricted to a subset of the real numbers, namely the computable numbers, and the functions, sequences and so on over this subset are restricted to the algorithmically definable ones. Aberth [1980, p. 4] concludes that

> [C]omputable analysis may be thought of as a subanalysis of real analysis. The two analyses differ but do not contradict each other.

Reverse mathematics aims to determine the non-computable set existence axioms necessary in order to prove theorems of ordinary mathematics, including analysis. We can therefore think of computable analysis as the other side of the coin, showing what can be done computably. While computable analysis

[22] Weihrauch [2000] proposes a somewhat different approach to computable analysis (the Type-2 Theory of Effectivity), parts of which somewhat resemble constructive analysis.

tends to deal more directly with analytical objects, rather than being restricted to natural numbers and sets thereof, the underlying principle of restricting attention to computably-definable objects is the same, as is the compatibility with (if not advocacy of) the assertion that every total function $f : \mathbb{N} \to \mathbb{N}$ is recursive. The degree to which RCA_0 is a faithful formalisation of computable analysis could therefore benefit from studying whether computable analysis can be formalised directly in a conservative higher-type system such as Kohlenbach [2005]'s.

All this being said, computable analysis is not a foundational programme in the same sense that constructive analysis is. It seems right to say that reverse mathematical results show what parts of classical analysis employ non-computable resources, and thus cannot be proved in computable analysis, but researchers in computable analysis are not committed to the kind of claims that constructivists are, namely that all of mathematics should be refounded on a computable basis. Nevertheless, reverse mathematics does seem to have a job to do in this context, regardless of whether we consider it a properly foundational one, although in terms of the computational foundations of analysis for physics, constraints based on feasibility seem more likely to have an impact than those based on uncomputability.

4.3 Partial realisations of Hilbert's programme

Hilbert's programme was to reduce infinitary mathematics to finitary mathematics. He viewed finitism as a secure foundation for mathematics, free of the paradoxes which arose from seemingly natural assumptions and normal mathematical reasoning about infinite collections. This reduction was to be accomplished by giving a finitary consistency proof for an infinitary system which, following Simpson [1988a], we can identify with Z_2. Hilbert thought that employing infinitary methods in mathematics, such as assuming the existence of infinite collections, could be viewed simply as a way to supplement our finitistic theories with ideal statements, analogous to ideal elements in algebra. Ideal statements are thus intended to be eliminable, at least in principle: the purpose of Hilbert's desired consistency proof was to show that we can use infinitary mathematics to get finitary results, and that those results are finitistically acceptable.

Gödel's second incompleteness theorem shows that there can be no such consistency proof, and thus that Hilbert's programme cannot be carried out in its entirety. Many authors even consider Gödel's theorems to have shown that Hilbert's programme is entirely bankrupt. While it certainly blocks the full realisation of the enterprise, Simpson [1988a] argues that the possibility of partial realisations remains. But since the consistency proof Hilbert sought is out of reach, the latter-day finitistic reductionist must find other ways to demonstrate that their uses of ideal statements are in principle eliminable. Instead of trying to prove the consistency of the infinitary system directly, finitistic reductions of infinitary systems can be carried out in a relativised way, following the template laid down by Kreisel [1968]. A comprehensive survey can be found in Feferman [1988], which I paraphrase here.

Suppose we have two theories T_1 (in a language $\mathcal{L}_1$) and T_2 (in $\mathcal{L}_2$), both of which contain primitive recursive arithmetic. Suppose also that we have a primitive recursive set of formulae $\Phi \subseteq Fml_{\mathcal{L}_1} \cap Fml_{\mathcal{L}_2}$ containing every closed equation $t_1 = t_2$. A proof-theoretic reduction of T_1 to T_2 which conserves Φ is a partial recursive function f which, given any proof from the axioms of T_1 of a sentence $\varphi \in \Phi$, produces a proof of φ from the axioms of T_2. If the existence of f can be proved in T_2, it then follows that T_2 proves (a formalisation of) the following conditional statement: "If T_2 is consistent then T_1 is consistent." For if T_1 proves that $0 = 1$, then f will transform any proof of $0 = 1$ in T_1 into a proof of $0 = 1$ in T_2.

Such a relative consistency proof will constitute a finitary reduction if the existence of f can be proved in a suitable finitary system. Clearly this is a requirement for the finitistic reductionist. Otherwise the result has a circular character unacceptable within a reductionist programme: it would amount to using ideal methods to show that ideal methods are acceptable. This is also why Hilbert wanted a finitary consistency proof for infinitary mathematics, since an infinitary proof would fail to appropriately reduce infinitary mathematics to finitary mathematics. Similarly, an infinitary proof of a conservativity theorem is insufficient to demonstrate the reducibility of an infinitary system to a finitary one. As Sieg [1985, p. 34] puts it, "[a conservativity theorem of this kind], if established by elementary means [i.e. finitary methods], is of obvious foundational significance as it gives a direct finitist justification for parts of mathematical practice."

If Hilbert had succeeded in providing a finitary consistency proof for infinitary mathematics then there would have been no need to mark out the boundary between finitary and infinitary methods with any precision, as the proof would have made use of methods which were clearly finitary in nature. Simpson's route to a partial realisation of Hilbert's programme requires that we formalise our conception of a finitary system, in order to obtain the conservation results that demonstrate that certain infinitary systems are finitistically reducible and partially realise Hilbert's programme. The formal system which Simpson selects is primitive recursive arithmetic or PRA. Tait [1981] argues that PRA is the correct formalisation of finitary mathematical reasoning, going so far as to say (p. 525) that

> We shall see that there is no question but that [primitive recursive] reasoning is finitist. The issue of our thesis will be whether all modes of finitist reasoning are primitive recursive.

Tait concludes that we can identify primitive recursive reasoning with finitist reasoning; this is now commonly known in the literature as *Tait's thesis*. Simpson [1988a, p. 352] concurs with Tait's thesis, writing that

> There seems to be a certain naturalness about PRA which supports Tait's conclusion. PRA is certainly finitistic and "logic-free", yet sufficiently powerful to accommodate all elementary reasoning about natural numbers and manipulations of finite strings of symbols. PRA seems to embody just that part of mathematics which remains if we excise all infinitistic concepts and modes of reasoning. For my purposes here I am going to accept Tait's identification of finitism with PRA.

The rest of Simpson's argument rests squarely on this identification of finitism with PRA: he does not offer any new considerations in support of Tait's thesis, instead simply accepting it and proceeding accordingly.

Fixing PRA as the finitary system to which infinitary systems must be reduced to, the next question is which infinitary systems are finitistically reducible to PRA. Simpson's answer is WKL_0, the system obtained by adding weak König's lemma ("Every infinite subtree of $2^{<\mathbb{N}}$ has an infinite path") to RCA_0. Friedman [1976, unpublished] used model-theoretic techniques to show that WKL_0 is Π^0_2 conservative over PRA; the proof can be found in Simpson [2009, §IX.3]. Subsequently Sieg [1985] gave a primitive recursive proof transformation which, given a proof of a Π^0_2 theorem φ in WKL_0, generates a proof of φ in PRA. Unlike Friedman's result this proof-theoretic derivation of the conservativity theorem is itself finitary in the appropriate way: it is provable within a finitary system and thus allows the reduction to go through. As the complexity of consistency statements is Π^0_1, if WKL_0 proves the consistency of PRA then so does PRA itself. By Gödel's second incompleteness theorem PRA would therefore be inconsistent. From this Simpson concludes that WKL_0 is finitistically reducible to PRA, and so the fragment of mathematical reasoning which one can carry out in WKL_0 is finitarily acceptable.

There are several aspects of Simpson's view that we might criticise. The first is his reliance on Tait's thesis, which has taken fire from many quarters. Schirn and Niebergall [2003] claim (p. 66) that "the identification of finitist mathematics with PRA is questionable, if not untenable". Broadly speaking such complaints fall into two camps: that PRA is too weak to encompass all of finitistic reasoning, and that it is too strong. Those in the former camp include Kreisel [1960], who concluded that finitary provability coincides with provability in PA. Detlefsen [1979] has argued that adding instances of the restricted ω-rule is also finitistically acceptable, although Detlefsen's position has in turn been criticised, for example by Ignjatović [1994]. Two proposals that fall into the latter camp are made by Ganea [2010]. From the broad spread of conclusions reached it is clear that what finitistic reasoning consists in is disputed, to say the least. Tait's arguments provide a robust defence of the thesis that primitive recursive arithmetic demarcates finitistic mathematical reasoning, and on this basis Simpson has presented a compelling foundational picture that should be taken seriously on its own merits.

This response also seems appropriate to the second criticism we shall consider, due to Sieg [1990], which amounts to the claim that Simpson's understanding of Hilbert is a misreading.

> Simpson considers the finitist reductionist program ... as Hilbert's program. This is inaccurate. Hilbert did not propose to redo all of mathematics with only finitist principles, but rather to justify—via finitist consistency proofs—the use of strong classical theories sufficient for the direct formalization of mathematical practice. If this particular reductionist program should be adorned with a name, then it seems appropriate to attach Kronecker's to it. Recall that on Hilbert's view the principles accepted by Kronecker coincided essentially with finitist ones, and Kronecker certainly insisted on using just those. Indeed, it would be highly interesting and quite possibly mathematically rewarding, if parts of Kronecker's work were to be analyzed within restricted axiomatic frameworks.

"A partial realisation of Kronecker's programme" does not have quite the same ring to it, but while issues of textual interpretation are important, they should not distract us from other salient issues, namely whether Simpson's finitistic reductionism is a substantial foundational programme worthy of proof-theoretic analysis. The answer must be that it is. The preceding criticisms do cast some doubt on the claim that WKL_0 constitutes a partial realisation of Hilbert's programme, but nevertheless, the reverse mathematics of WKL_0 clearly make a foundational contribution, insofar as they demonstrate what fragment of ordinary mathematics can be recovered within this framework, whatever we choose to call it.

However, even if we take Tait's thesis for granted, Simpson's argument does not in any way pick out WKL_0 as the unique formal counterpart of finitistic reductionism. Brown and Simpson [1993] present a system they call WKL_0^+, which extends WKL_0 with a strong formal version (BCT) of the Baire Category Theorem. They prove, using a forcing argument, that WKL_0^+ is Π^1_1 conservative over RCA_0, and therefore by a result of Parsons [1970], Π^0_2 conservative over PRA. Since (BCT) is a scheme involving formulas of arbitrary complexity, Sieg's methods are inapplicable. However, by formalising the forcing argument in RCA_0, Avigad [1996] effectivizes the conservativity theorem and thus demonstrates that WKL_0^+ is also finitistically reducible. So while WKL_0 is, modulo Tait's thesis, a finitarily reducible system, it is but one partial realisation of Hilbert's programme. WKL_0^+ is demonstrably another, and indeed a stronger one, since it satisfies the same criteria of finitistic reducibility whilst properly extending WKL_0.

One might think that this undermines Simpson's claim that the Big Five subsystems of second order arithmetic correspond to existing foundational programmes, but this is not a fair reading of Simpson's position: he does not claim that these systems are the unique formal correlates of these foundational approaches (henceforth, we shall call this the *uniqueness claim*). It is consistent with his position that there are a variety of infinitary yet finitistically reducible systems. This being said, the stress he places on these particular correspondences makes it reasonable to suppose that he may, in fact, accept some form of the uniqueness claim.

Moreover, the striking results of reverse mathematics do give rise to the expectation that there is something to the uniqueness claim. The vast majority of ordinary mathematical theorems studied to date have been found to either be provable in the base theory RCA_0, or to be equivalent to one or other of the Big Five. Simpson [2010, p. 115] estimates that "several hundreds [of theorems] at least" have been thus classified. This seems to constitute evidence of a quasi-empirical form that these systems are natural stopping points. If we provisionally accept some form of Simpson's claim that each of these systems can be justified on the basis of an antecedently understood foundational programme, and also that each system cannot be justified on the basis of the foundational principles that justify the system below it in the ordering (for example, ACA_0 can be justified by predicativism but not Simpson's partial realisation of Hilbert's programme), then it would be reasonable to expect all of these systems to be the strongest ones justifiable on the basis of those foundational programmes. It is this maximality expectation that gives rise to the uniqueness claim.

As we have seen, however, WKL_0 does not appear to be the strongest system justified on the basis of Simpson's finitistic reductionism, and as we shall see in the rest of this section, this expectation is also violated elsewhere. The moral seems to be that proof-theoretically natural closure points do not always align cleanly with justificatory closure points—or if they do, then we have not yet identified the sources of justification of these axiom systems in a sufficiently fine-grained way.

4.4 Predicativism and predicative reductionism

ACA_0 has a close connection to predicativism of the form associated with Weyl and Feferman. Feferman [2005, p. 599] writes that "Weyl accepted that each subset of $\mathbb{N}$ of the form $\{ n \in \mathbb{N} \mid A(n) \}$ exists, where A is an arithmetical formula (i.e., one that contains no quantifiers ranging over sets, only over natural numbers)." This aligns perfectly with the arithmetical comprehension scheme, which precisely asserts that those sets exist which are definable by arithmetical formulas. Indeed, when Feferman [2005, p. 610] discusses positive developments in the mathematical reach of predicativity, he writes that "The primarily relevant system for the positive work on predicative mathematics in [reverse mathematics] is ACA_0".

Predicativity given the natural numbers can be extended beyond ACA_0 in a natural and obvious way, by allowing comprehension principles in which the quantifiers range over sets which have already been determined to exist on predicative grounds. This process can be iterated through ω-many stages and beyond, giving rise to the *ramified analytical hierarchy* of sets of natural numbers. Corresponding systems of ramified analysis $\mathbf{RA}_\alpha$ are then defined in terms of comprehension principles which express the closure conditions that apply at each stage. The following rough sketch is merely intended to give a sense of how predicative reductionism extends the version of predicativism associated with ACA_0 to reach the greater proof-theoretic strength of ATR_0. Readers interested in understanding the programme in more detail are directed to Feferman [2005] who explains its historical and technical development in some detail. Predicative reductionism and reverse mathematics are also discussed by

Simpson [1985, pp. 152–156].

A formal system T is *predicatively reducible* if it is proof-theoretically reducible to one of the systems $\mathbf{RA}_\alpha$ such that $\alpha < \Gamma_0$, the Feferman–Schütte ordinal (where proof-theoretic reducibility is defined as in §4.3). ACA_0 is predicatively reducible in just this sense. If, on the other hand, T is proof-theoretically reducible to the union of all the predicative systems of ramified analysis $\mathbf{RA}_{\Gamma_0} = \bigcup_{\alpha<\Gamma_0} \mathbf{RA}_\alpha$ then we say that T is *locally predicatively reducible*. By a theorem of Friedman, McAloon, and Simpson [1982], ATR_0 is locally predicatively reducible. Moreover, ATR_0 is Π^1_1 conservative over $\mathbf{RA}_{\Gamma_0}$. So not only does ATR_0 agree with the predicative part of ramified analysis about arithmetical truth, it also proves the same theorems about the arithmetical properties of all real numbers.

The formal system ATR_0 consists of ACA_0 plus a scheme of *arithmetical transfinite recursion*. This states that the arithmetical operations can be iterated, starting from any set $X \subseteq \mathbb{N}$, along any countable wellordering. For a full formal definition see Simpson [2009, §V.2]. ATR_0 can therefore prove the consistency of ACA_0, by iterating the Turing jump operator ω-many times and constructing the code for a countable ω-model of ACA_0. As the reverse mathematics programme has shown, there are many theorems not provable within ACA_0 that ATR_0 does prove, so predicative reductionism is a significant strengthening of the predicative outlook, albeit one still operating within the framework that informed the acceptance of ACA_0 as a predicative system. Moreover, ATR_0 sits at the outer limits of predicativity, since its proof-theoretic ordinal is Γ_0—the "ordinal of predicativity", as determined by Feferman [1964] and Schütte [1964, 1965].

4.5 Impredicative systems

The foundational role of the impredicative system Π^1_1-CA_0 is less clear cut. By results of Feferman [1970], Π^1_1-CA_0 can be proof-theoretically reduced to the theory of iterated inductive definitions $\mathsf{ID}_{<\omega}$. This system can in turn be reduced to an intuitionistic version of itself, $\mathsf{ID}^i_{<\omega}(\mathcal{O})$, by an extension of the double-negation translation. However, this is a property which Π^1_1-CA_0 has in common with other impredicative subsystems of analysis such as Σ^1_2-AC, which by Friedman [1970] is reducible to the theory $\mathsf{ID}_{<\varepsilon_0}$. Full details of these reductions appear in Feferman and Sieg [1981].

We might reasonably wonder what is achieved by such reductions. The allegedly constructive character of these intuitionistic theories of iterated inductive definitions must be demonstrated, in order for us to conclude that these reductions put impredicative subsystems of analysis on a more secure epistemic footing. Sieg [1984] argues for this conclusion as follows (pp. 187–8):

> The theories are based on intuitionistic logic; the objects in their intended models are obtained by construction; the definition- and proof-principles which are admitted in the theories follow that construction. The objects, i.e., the constructive ordinals, are furthermore of a very special character. They reflect their buildup according to the generating clauses of their definition in a direct and locally effective way. Viewing the clauses as inference rules, the constructive ordinals are infinitary derivations and show that they fall under their definition. All of this indicates that the theories for ordinals are constructively justified and thus provide a constructive foundation for the classical theories which are reducible to them.

Sieg continues (p. 188) by saying that

> From a broader perspective, I see these investigations as part of an attempt to take the concept of iteration or inductive definition as basic for analyzing that section of mathematics which lends itself to an arithmetic, constructive treatment.

Such constructive foundations for subsystems of analysis form an important part of the generalised Hilbert programme, as articulated in Sieg [1988]. They do not, however, in any way satisfy the uniqueness claim discussed in §4.3. In particular they offer no defence of the particular importance of Π^1_1-CA_0. Thus while reverse mathematics has much to say about how much ordinary mathematics can be developed in this system, and thus how much can be constructively justified on the basis of the particular considerations that lead to the acceptance of the principles of the system $\mathsf{ID}^i_{<\omega}(\mathcal{O})$, there does not seem to be any principled reason lurking in the background as to why we should stop here if we can possibly go on. In other words, the connection that Simpson [2009, p. 43] makes between Π^1_1-CA_0 and the work of Buchholz, Feferman, Pohlers, and Sieg [1981] on iterated inductive definitions is real enough, but it is more subtle than the simple idea presented in the first section of this chapter, namely that in proving reversals we thereby show what mathematics can and cannot be developed in the formal counterparts of particular philosoophically-motivated foundations for mathematics.

Moreover, reverse mathematical results have less direct relevance for foundations which can accommodate impredicative principles such as Π^1_1 comprehension, partly because most ordinary mathematical theorems studied to date have turned out to be proof-theoretically weaker. That a theorem τ is provable in Π^1_1-CA_0 is of course important from the point of view of recapturing ordinary mathematics on the basis of constructive principles, but while interesting, any reversal to Π^1_1 comprehension is more important from the perspective of weaker foundational perspectives, since the equivalence demonstrates the theorem's unprovability in their framework. What would be truly interesting from the viewpoint of constructively-justifiable impredicative analysis would be reversals from ordinary mathematical theorems to much stronger impredicative subsystems of analysis that are yet to be placed on a constructive footing.[23]

[23] One value that reversals do have for the generalised Hilbert programme is letting us know that the work was worth it. For example, the equivalence of a substantial fragment of descriptive set theory to Π^1_1 comprehension means that reducing Π^1_1-CA_0 to a theory based on constructively acceptable principles allows us to place this part of mathematics on a sound footing, and that we would not have been able to do so otherwise.

5

Computational reverse mathematics

5.1 Shore's programme

However much it may borrow from other areas of mathematical logic, reverse mathematics is ultimately a proof-theoretic endeavour. Given a theorem of ordinary mathematics, the reverse mathematician seeks to find a subsystem of Z_2 that is equivalent over a weak base theory to the theorem concerned. She thereby finds the proof-theoretic strength of the theorem. Rooted in niceties of formal systems such as axiom schemes and complexity hierarchies of formulae, this approach may seem awkward and even unnatural to mathematicians in more mainstream fields. As number theorist Barry Mazur explains [Mazur 2008, p. 224—emphasis in original],

> when it comes to a crisis of rigorous argument, the open secret is that, for the most part, mathematicians who are not focussed on the architecture of formal systems per se, mathematicians who are *consumers* rather than *providers*, somehow achieve a sense of utterly firm conviction in their mathematical doings, without actually going through the exercise of translating their particular argumentation into a brand-name formal system.

Turning to the specific case of the strength of mathematical theorems, Richard Shore contends that most mathematicians do not approach this task from the viewpoint of reverse mathematics [Shore 2010, p. 381]:

> While they may concern themselves with (or attempt to avoid) the axiom of choice or transfinite recursion, they certainly do not think about (nor care), for example, how much induction is used in any particular proof.

Shore goes on to argue that adopting a computational approach to reverse mathematics would solve this exegetical problem, providing a natural way for mathematicians to understand the motivations and results of reverse mathematics. Whether algorithmic and construction-oriented explanations are more natural to the main body of mathematicians is an interesting question, but not one I shall attend to here. In labelling his framework as a strain of reverse mathematics, Shore invites comparisons with the traditional variety. The major task ahead of us is to examine whether Shore's programme offers a comparable or superior way of achieving one of the foundational goals set out by Simpson: carrying out foundational analysis.

Instead of formal provability, the fundamental concept of Shore's framework is *computable entailment*. As well as motivating computational reverse mathematics by examining the close links between computational principles and arithmetic, and looking at how tools from recursion theory are already employed by working reverse mathematicians, this section explains Shore's entailment relation. It quickly becomes clear that it collapses many distinctions present in traditional reverse mathematics, giving a rather different picture of the relationships between theorems of ordinary mathematics and the computational and combinatorial principles required to prove them.

A computational account of reverse mathematics can be considered plausible only if mathematical principles have computational content. At least in the case of arithmetic it is clear that this is true, as demonstrated by the pioneering results of Gödel, Church, Turing, Post, Kleene and Rosser in the 1930s. Recursion theory holds an important status in reverse mathematics, both in virtue of its relationship to subsystems of reverse mathematics and because it provides a battery of tools for proving reverse mathematical results. It is these principles and techniques which Shore appeals to when constructing his account of computational reverse mathematics.

The major subsystems of second order arithmetic correspond to classical principles from recursion theory. As well as shedding light on the model theory of these systems, these connections give us the basis for Shore's computational reverse mathematics. The foundation of these correspondences lies in the notion of an ω*-model*. An ω-model is one whose first order part consists of the standard natural numbers $\omega = \{ 0, 1, 2, \dots \}$, and whose arithmetical vocabulary is interpreted in the standard way, with a second order part $\mathcal{S} \subseteq \mathcal{P}(\omega)$. ω-models are thus uniquely distinguished by their second order parts, and henceforth we shall be sloppy and identify ω-models with their second order parts wherever no ambiguity is possible.

The sets of natural numbers lying in $\mathcal{S}$ determine which systems the ω-model satisfies. Since subsystems of second order arithmetic are principally characterised by their comprehension schemes, the more definable sets $\mathcal{S}$ contains the stronger the systems it can satisfy. If $\mathcal{S}$ is closed under Δ^0_1 definability then it will satisfy RCA_0. The following fact demonstrates the relationship between definability and relative computability, and will prove useful in what follows.

Definition 5.1.1. Let $X, Y \subseteq \mathbb{N}$. The *recursive join of X and Y* is given by

$$X \oplus Y = \{\, 2x \mid x \in X \,\} \cup \{\, 2y + 1 \mid y \in Y \,\}. \tag{5.1}$$

Definition 5.1.2. Let $\mathcal{C}$ be a nonempty subset of $\mathcal{P}(\omega)$ closed under Turing reducibility and recursive joins. Then we call $\mathcal{C}$ a *Turing ideal.*

Fact 5.1.3. *An ω-model M is a model of RCA_0 iff its second order part is a Turing ideal.*

Similar closure conditions apply to the ω-models of the other main subsystems of second order arithmetic. ω-models of ACA_0 are Turing ideals, since RCA_0 is a subtheory of ACA_0, but these models are also closed under the Turing jump operator, while those of Π^1_1-CA_0 are closed under the hyperjump. Closure under recursion-theoretic relations also characterises the ω-models of the intermediate systems WKL_0 and ATR_0. The ω-models of WKL_0 are related to the Jockush–Soare low basis theorem [Jockusch and Soare 1972]. The ω-models of ATR_0 are closed under hyperarithmetic reducibility, although the story here is more subtle, since the class of hyperarithmetic sets HYP is so closed, but is not an ω-model of ATR_0 (although it is the intersection of all such models); see §VIII.4 and §VIII.6 of Simpson [2009]. The Big Five thus correspond closely to a hierarchy of computational principles of increasing power.

Computability theory also provides important tools for the practising reverse mathematician. A common application is using Turing ideals to prove nonimplications between statements or theories. They form a natural class of models where the Turing reducibility relation behaves as we expect, so it is a good setting in which to find countermodels. The procedure is particularly straightforward when the sentences in question are Π^1_2; Shore [2010, p. 384] gives a detailed explanation. Since many important mathematical theorems such as the Ascoli lemma and Ramsey's theorem are Π^1_2, the technique is widely applicable. For instance, to show that weak König's lemma does not imply arithmetical comprehension, we note that by the Jockush–Soare low basis theorem there is an ω-model M of WKL_0 in which all sets are low. Such a model will not contain $0'$, and thus $M \not\models \mathsf{ACA}_0$, since ACA_0 proves the existence of the Turing jump.

Shore proposes taking this use of computability theory a step further and basing a new approach to reverse mathematical analysis on recursion theory, rather than proof theory. In place of the usual relations employed in reverse mathematics—provability and logical equivalence over a base theory—he offers the notions of computable entailment and computable equivalence.

Definition 5.1.4. Let $\mathcal{C}$ be a Turing ideal, and let φ be a sentence of second order arithmetic. $\mathcal{C}$ *computably satisfies* φ if φ is true in the ω-model whose second order part consists of $\mathcal{C}$. A sentence ψ *computably entails* φ, $\psi \models_c \varphi$, if every Turing ideal $\mathcal{C}$ satisfying ψ also satisfies φ. Two sentences ψ and φ are *computably equivalent*, $\psi \equiv_c \varphi$, if each computably entails the other. These definitions extend to theories in the standard way.

Computable entailment removes any need for an explicit base theory: this role is instead played by the restriction of the class of models under consideration to ω-models whose second order parts are Turing ideals. As fact 5.1.3 shows, the ω-models of RCA_0 are precisely those models, so the base theory has not disappeared but merely manifested itself in a different way. Furthermore, failures of computable entailment are stronger than failures of logical implication over RCA_0, since the former entails the latter, but not vice versa. Conversely, a proof of computable entailment is weaker than logical implication over RCA_0. Considered as a variant of reverse mathematics, Shore's approach in his [2010] and [2013] is revisionary: computable entailment and equivalence are not coextensional with their proof-theoretic counterparts. It is not merely an alternative way of conceiving of the role and significance of traditional reverse mathematics, but a substantially different project, albeit a closely related one.

Shore does not offer computational reverse mathematics as a way to carry out foundational analysis; he has very different methodological goals in mind. But given its advantages over the classical way of doing business, namely that we can use recursion-theoretic machinery directly without too much concern over niceties such as the amount of induction available, it seems reasonable to wonder whether his framework can contribute to the analysis of foundational programmes in the same way as classical reverse mathematics.

In §2.7 we discussed some considerations in favour of Shore's approach.[24] On the pragmatic side, restricting the amount of induction available in the base theory is occasionally problematic, since some equivalences cannot be proved without increasing the strength of the induction axiom used. Restricting

[24]The objections raised there are relevant to a different set of issues to those considered here, so we will not go over them again.

our attention to ω-models is the ideal limit of this process, as the standard natural numbers satisfy the induction scheme when formulated in second order arithmetic as well as all higher types. By fixing the first-order part of the model we remove many complications and make reverse mathematics a more purely computability-theoretic endeavour.

Moreover, the core subject matter of reverse mathematics consists in ordinary mathematical statements about actually infinite objects such as real numbers, complete separable metric spaces and countable groups and fields. These statements are invariably Π^1_2 (and, rarely, Π^1_3) rather than arithmetical. By fixing the first order part of the model and allowing the second order part to vary across different Turing ideals, we remove any indeterminacy about the arithmetical world from our framework and focus on the central issue, namely the equivalences between ordinary mathematical statements and computability-theoretic closure conditions on $\mathcal{P}(\mathbb{N})$.

Shore's proposal that we restrict our attention to ω-models of RCA_0 is, by the Henkin/Orey completeness theorem for ω-logic [Orey 1956, Henkin 1954], extensionally equivalent to adding the ω-rule to RCA_0. The ω-rule is an infinitary rule of inference that, from the infinite set of premises $\varphi(0), \varphi(1), \ldots, \varphi(\overline{n})$, $\ldots$, one may infer the universal statement $\forall n \varphi(n)$. The infinitary "proofs" in this system are represented by wellfounded, countably branching trees. The classical ω-rule is, by a result of Lopez-Escobar [1967] and Takahashi [1970], extensionally equivalent in second order arithmetic to the restricted ω-rule introduced by Shoenfield [1959]. Second order arithmetic with the ω-rule is complete for Π^1_1 sentences, but not for Σ^1_1 sentences [Rosser 1937].

5.2 Computable entailment and justification

Computable entailment collapses many distinctions present under the usual classical entailment relation, and thus the equivalence classes obtained under the computable equivalence relation are very different from those given by provable equivalence over RCA_0. For instance, the standard natural numbers satisfy the induction scheme for all predicates in the language of second order arithmetic. As a result, systems with only restricted induction and their counterparts with the full induction scheme are computably equivalent. The presence of full induction is indicated by the absence of the '0' subscript in the system's name: RCA is RCA_0 but with full induction, WKL is WKL_0 with full induction, and so on. In all cases, the system with full induction has precisely the same ω-models as its counterpart with restricted induction, and thus they are computably equivalent.

This presents a problem given the connections between the Big Five and existing philosophically-motivated programmes in the foundations of mathematics. At least in some cases these subsystems are formalisations in second order arithmetic of those foundational programmes, but it is by no means obvious that the same is true for other axiom systems which are computably equivalent to them. ACA_0 is a predicative system, but the mere fact that ACA is computably equivalent to it should not compel us to believe that ACA is similarly predicatively acceptable.

Another way to understand this point is by considering that a key property of any entailment relation is preserving justification: if we are justified in accepting the antecedent then we are justified in accepting the consequent. For computational reverse mathematics to be capable of the foundational analysis outlined earlier, we must show that computable entailment preserves justification just as deductive entailment does. Crucially, we must show that given any foundational programme that we wish to analyse by proving reverse mathematical results, those results will be justified on the conception of justification internal to the foundational programme itself. If computable entailment fails to satisfy this requirement then proponents of such foundational programmes will be unmoved by any arguments drawn from computational reverse mathematics, as they will reject the underlying assumption necessary to proving the results involved. In other words, the crux of the issue is not whether computable entailment preserves justification on some particular account of the epistemology of mathematics, but whether it respects the justificatory structure of the foundational programmes being analysed.

In the previous chapter we examined Simpson [1988a]'s claim that the Π^0_2 conservativity of WKL_0 over primitive recursive arithmetic means that the former, infinitary system can be reduced to the latter, finitary one, and that this constitutes a partial realisation of Hilbert's programme. There are reasons to question whether Simpson's interpretation of Hilbert is correct, and plenty of debate to be had over whether this is in fact a good foundation for mathematics. Nevertheless, the finitistic reductionism that Simpson proposes is nonetheless a foundational enterprise worthy of consideration. While it has its detractors, Tait's thesis that finitistic provability is to be identified with provability in PRA has also gained widespread support, and the other crucial element of Simpson's argument—the finitistic reduction of WKL_0 to PRA—is not in doubt. For these reasons we should not attach undue importance to the name of Hilbert, but instead assess Simpson's approach on its own merits.

One part of such an assessment consists of the use of reverse mathematical methods to determine the parts of ordinary mathematics that can be developed within this foundational framework. Our system of reverse mathematics should therefore be able to analyse Simpson's finitistic reductionism, and as argued above, that analysis should respect its justificatory structure. With this concern in mind, the crucial question is whether or not Simpson's partial realisation of Hilbert's programme can be extended from WKL_0 to include all systems T that are computably equivalent to WKL_0. Only if this is the case can we conclude that Shore's computational reverse mathematics respects the justificatory structure of Simpson's finitistic reductionism.

One system that is computably equivalent to WKL_0 is the system WKL. As mentioned earlier, this system augments WKL_0 with the full induction scheme (1.13). If computable entailment is to preserve justification for the finitist, then WKL must also be finitistically reducible. But the presence of the full induction scheme means that, as we shall see below, WKL proves the consistency of PRA. Therefore, it is not finitistically reducible to PRA, since the canonical consistency statement $\mathrm{Con}\,\mathrm{PRA}$ is a Π^0_1 statement that PRA does not (if it is, in fact, consistent) prove. In other words, it rules out the possibility of a finitistic reduction of the sort delivered by Sieg for WKL_0, and thus rules out the possibility that WKL is a finitistically reducible system.

Recall that $I\Sigma_n$ is the fragment of Peano arithmetic obtained by restricting the induction scheme to Σ^0_n formulae. The following is a standard result in the literature on first-order arithmetic. A full proof can be found in Hájek and Pudlák [1993, §I.4].

Fact 5.2.1. *$I\Sigma_{n+1}$ proves the consistency of $I\Sigma_n$.*

Corollary 5.2.2.

1. *$I\Sigma_1$, PRA, RCA_0 and WKL_0 are equiconsistent.*

2. *WKL proves the consistency of the systems given in (1).*

3. *WKL is not Π^0_1 conservative over the systems given in (1).*

Proof. $I\Sigma_1$ is Π^0_2 conservative over PRA [Parsons 1970]; the first order part of RCA_0 is $I\Sigma_1$ (that is, they prove the same sentences in the language L_1 of first order arithmetic); and WKL_0 is Π^1_1 conservative over RCA_0 (this is a result of Leo Harrington; a proof appears in Simpson [2009, §IX.2]). Consequently any Π^0_2 statement provable in WKL_0 (or RCA_0 or $I\Sigma_1$) is also provable in PRA. Since the canonical consistency statements for PRA, $I\Sigma_1$ and WKL_0 are Π^0_1, any system proving the consistency of one of these systems proves the consistency of all the others.

By fact 5.2.1, $I\Sigma_2$ proves the consistency of $I\Sigma_1$ and hence the consistency of all the systems listed in (1). WKL extends $I\Sigma_2$ and thus proves all the theorems it does. Finally, by the complexity of consistency statements, WKL cannot be Π^0_1 conservative over any of the systems listed in (1). □

The methods of infinitary mathematics are justified, on Simpson's reading of Hilbert's view, only to the extent that they are reducible to finitary ones. This seems to rule out WKL as a partial realisation of Hilbert's programme quite straightforwardly. But if computable entailment preserves justification, then we are justified in accepting WKL if and only if we accept WKL_0, as they are computably equivalent. If this is not the case then computable equivalence seems to have failed as a way to analyse the mathematical resources required to derive theorems of ordinary mathematics, since it leads to underdetermination: we are no longer certain, given some theorem φ, whether it is acceptable to the finitistic reductionist if we know only that it is computably entailed by WKL_0. To resolve this underdetermination we must show that φ follows from WKL_0 using only resources acceptable to the finitistic reductionist—but since these resources are simply the axioms of a finitistically reducible system and the laws of classical logic, this amounts to simply proving the result in WKL_0, and we are no longer working in Shore's framework, where all that is necessary to show that one principle follows from another is to demonstrate that it is true in every ω-model of the first.

This being the case, we have at least one situation in which computational reverse mathematics is not sufficient to carry out a task in reverse mathematics of significant philosophical interest and importance. The computable entailment relation does not always preserve the justificatory structure of foundational theories, and hence Shore's framework thus cannot be used to conduct the kind of foundational analysis articulated in the previous chapter.

5.3 The complexity of computable entailment

We now turn to a different but related issue with the computable entailment relation: its recursion-theoretic complexity. As we know from Church and Turing's negative answer to the Entscheidungsproblem, the classical provability relation is uncomputable. Indeed, the set of provable consequences of a theory like Peano arithmetic is a quintessential example of a recursively enumerable set that is not recursive. Consequently, while there is no general method for determining whether or not a sentence φ in the language of arithmetic is provable in RCA_0, there is a Turing machine which enumerates the provable consequences of RCA_0, amongst which are the equivalences of classical reverse mathematics.

Semantic relations such as truth tend to be far more complex than syntactic relations such as provability, since they are—usually ineliminably—infinitary in nature. I say "usually" since the completeness theorem for classical first order logic gives us an important counterexample. As

$$T \models \varphi \;\Leftrightarrow\; T \vdash \varphi \tag{5.2}$$

for theories T and sentences φ, we can enumerate the model-theoretic consequences of a theory by enumerating its provable consequences, reducing a complex semantic relation to a finitary one. The same does not hold for computable entailment. Not only is it not recursive, but it is not even arithmetical. As a prelude to demonstrating this, we give a slightly revised definition of computable entailment, generalised to accommodate parameters.

Definition 5.3.1. For any set $X \subseteq \mathbb{N}$, and sentence φ in the language L_2 expanded with a constant symbol for X, we say that φ *is X-computably entailed*, in symbols $\models_c^X \varphi$, iff for all Turing ideals M such that $X \in M$, $M \models \varphi$.

At first glance this may appear less general than the earlier definition, but by the definition of the satisfaction relation, $(\varphi \models_c^X \psi)$ iff $\models_c^X (\varphi \to \psi)$, and the new definition is simpler to work with in the current context. Fixing a recursive, bijective Gödel coding of sentences of second order arithmetic, we represent the computable entailment relation by the set of Gödel codes for sentences which are computably entailed. For any $X \subseteq \mathbb{N}$, let

$$C(X) = \{ \ulcorner\varphi\urcorner \mid \models_c^X \varphi \} \tag{5.3}$$

where φ is an L_2-sentence which may contain a constant $\overline{X}$ denoting X. The parameter-free version of $C(X)$ we denote simply C. Observing that the definition of computable entailment quantifies over ω-models, we can see that C contains all the sentences of True Arithmetic, the first order theory of the natural numbers. True Arithmetic is not arithmetically definable, as this would contradict Tarski's theorem. So computable entailment cannot be arithmetical either.

A stronger lower bound for the complexity of computable entailment can be found by noting that arithmetical properties of reals are absolute to all ω-models, and thus that all Π^1_1 sets of natural numbers are 1-reducible to C. We can thus precisely characterise its complexity as Π^1_1-complete, by showing that C can be captured by a Π^1_1 definition. This theorem is essentially a classical one due to Grzegorczyk, Mostowski, and Ryll-Nardzewski [1958, §3.4, pp. 386–7]. Their result was proved for the second order functional calculus with the ω-rule, which they refer to as A_ω. We can understand this in the terminology of the present work as the following result: the set of Gödel numbers of L_2-sentences true in every ω-model of second order arithmetic Z_2 is a Π^1_1-complete set. The proof presented below is due to Carl Mummert [Mummert 2012], who strengthens the classical theorem by proving it for ω-models of RCA_0 rather than full Z_2. By introducing the notion of X-computable entailment I further generalise the result to include set parameters, although Mummert's proof needs only cosmetic alterations to accommodate this generalisation.

Theorem 5.3.2. *For any set parameter $X \subseteq \omega$, the computable entailment relation $C(X)$ is $\Pi^1_1(X)$-complete.*

We shall need the following standard definitions from recursion theory. For more background the reader should consult a reference work such as Rogers [1967], Soare [1987], or the elegant and accessible presentation of Ash and Knight [2000].

Definition 5.3.3. For sets $X, Y \subseteq \omega$, X *is many-one reducible to* Y, $X \leq_m Y$, just in case there is a total recursive function f such that for all $m \in \omega$,

$$m \in X \Leftrightarrow f(m) \in Y. \tag{5.4}$$

If f is injective then X *is* 1*-reducible to* Y, $X \leq_1 Y$, and if f is a bijection then X and Y are 1*-equivalent*.

Definition 5.3.4. Let $\mathcal{X} \subseteq \mathcal{P}(\omega)$. A set $X \subseteq \omega$ is *complete for* $\mathcal{X}$ iff $X \in \mathcal{X}$ and $Y \leq_1 X$ for every $Y \in \mathcal{X}$.

Lemma 5.3.5. *For any set parameter $X \subseteq \omega$, every $\Pi^1_1(X)$ set A is* 1*-reducible to $C(X)$.*

Proof. Let $\varphi(m_1, X_1)$ be a Π^1_1 formula. We refer to $(\omega, \mathcal{P}(\omega))$ as the *full model*.

Claim: For any $n \in \omega$ and $X \subseteq \omega$, $\varphi(n, X)$ is true in the full model iff it's true in all Turing ideals containing X.

($\Leftarrow$) The full model is a Turing ideal containing X, so if $\varphi(n, X)$ is false in the full model then it's false in that ideal.

($\Rightarrow$) Assume without loss of generality that $\varphi(n, X) \equiv \forall Y \psi(n, X, Y)$ where ψ is arithmetical. Suppose there is a Turing ideal $\mathcal{C}$ containing X such that $\mathcal{C} \not\models \varphi(X)$. Then there is some counterexample $B \in \mathcal{C}$ such that $\mathcal{C} \not\models \psi(X, B)$. Since the interpretation of the first order quantifiers and nonlogical symbols are the same in all ω-models, such a B will remain a counterexample in the full model.

This completes the proof of the claim.

Given $\varphi(m_1, X_1)$ as above, let $A = \{ n \in \omega \mid \varphi(n, X) \}$. Define the function $f_A : \omega \to \omega$ as $f_A(n) = \ulcorner\varphi(\overline{n}, X)\urcorner$. This function is recursive and injective, since if $a \neq b$ then $\ulcorner\varphi(\overline{a}, X)\urcorner \neq \ulcorner\varphi(\overline{b}, X)\urcorner$ by the properties of the Gödel coding. Finally by the claim above and the fact that $\varphi(m_1, X_1)$ is Π^1_1, $n \in A \leftrightarrow \varphi(n, X) \leftrightarrow \ulcorner\varphi(\overline{n}, X)\urcorner = f_A(n) \in C(X)$. $\square$

Having shown that C is Π^1_1-hard, i.e. that all sets $A \in \Pi^1_1$ are 1-reducible to it, we shall show that C is itself Π^1_1 and thus is Π^1_1-complete. In doing so we shall lean on the following definition which shows how a set can code a countable Turing ideal. A countable coded ω-model is a set W which codes countable sequence of sets $\langle (W)_n \mid n \in \mathbb{N} \rangle$ where $(W)_n = \{ i \mid (i, n) \in W \}$. For a full definition of countable coded ω-models see Simpson [2009, §VII.2].

Definition 5.3.6. Suppose $W \subseteq \mathbb{N}$ is a set coding the countable model M and $X \subseteq \mathbb{N}$. W *codes a countable Turing ideal containing* X iff

(i) For every m, n, there exists a k such that $(W)_k = (W)_m \oplus (W)_n$;

(ii) For every m, if $Y \leq_{\mathbf{T}} (W)_m$ then there exists a k such that $(W)_k = Y$;

(iii) There exists some k such that $(W)_k = X$.

Lemma 5.3.7. *Let $X, W \subseteq \mathbb{N}$. The predicate "$W$ codes a countable Turing ideal containing X" is arithmetical.*

Proof. Throughout we use the countable coded ω-model W as a parameter. The following formula is an analogue of condition (i) of definition 5.3.6.

$$\begin{aligned} \forall m \forall n \exists k \forall x \forall y [\, & x \in (W)_m \wedge y \in (W)_n \\ & \leftrightarrow 2x \in (W)_k \wedge 2y + 1 \in (W)_k \,]. \end{aligned} \tag{5.5}$$

For (ii), let $\pi(e, n, Y)$ be a universal lightface Π^0_1 formula with the given free variables. The existence of such formulae is provable in RCA_0; a definition is provided in Simpson [2009, definition VII.1.3, p. 244]. They play the role of universal Turing machines.

$$\begin{aligned} \forall m \forall e_0 \forall e_1 [\, & \forall n (\pi(e_0, n, (W)_m) \leftrightarrow \neg \pi(e_1, n, (W)_m)) \\ & \to \exists k \forall n (n \in (W)_k \leftrightarrow \pi(e_0, n, (W)_m)) \,]. \end{aligned} \tag{5.6}$$

Finally we add condition (iii) that X is an element of the Turing ideal coded by W,

$$\exists k \forall n (n \in X \leftrightarrow n \in (W)_k). \tag{5.7}$$

One can (tediously) verify that these conditions hold of W if and only if the ω-model coded by W is a Turing ideal containing X. $\square$

Lemma 5.3.8. *For any set parameter $X \subseteq \mathbb{N}$, if an $\mathrm{L}_2(X)$-sentence φ is false in any Turing ideal containing X, then it is false in a countable Turing ideal containing X.*

Proof. Let M be a Turing ideal containing X, and assume that $M \models \neg\varphi$. By the downwards Löwenheim–Skolem theorem, M has a countable ω-submodel $M' \subseteq_\omega M$ such that $X \in M'$. M' is a Turing ideal, as this property is definable by an $\mathrm{L}_2(X)$ sentence which is true in M, and thus in M' by elementarity. Finally, φ is false in M', again by elementarity. $\square$

Proof of theorem 5.3.2. Fix a set parameter X. By lemma 5.3.5, $C(X)$ 1-reduces every $\Pi^1_1(X)$ set. It only remains to show that $C(X)$ is itself a $\Pi^1_1(X)$ set.

Let $C^\dagger(X)$ be the set of Gödel codes of L_2-sentences φ such that every countable Turing ideal containing X satisfies φ. Lemma 5.3.8 shows that any sentence φ of second order arithmetic is satisfied by every Turing ideal containing X iff it's satisfied by every countable Turing ideal containing X. So $\ulcorner\varphi\urcorner \in C(X) \Leftrightarrow \ulcorner\varphi\urcorner \in C^\dagger(X)$. Thus by proving that $C^\dagger(X)$ is a $\Pi^1_1(X)$ set, we show that $C(X)$ is also $\Pi^1_1(X)$.

The relation $\ulcorner\varphi\urcorner \in C^\dagger(X)$ can be defined in second order arithmetic as:

$$(\forall \text{ countable Turing ideals } M)(X \in M \to M \models \varphi) \tag{5.8}$$

To see that this is equivalent to a Π^1_1 formula, we note the following. Firstly, by lemma 5.3.7, the predicate "W codes a countable Turing ideal M" is arithmetical. Secondly, $M \models \varphi$ means "There exists a valuation function $f : \mathrm{Sub}_M(\varphi) \to \{0, 1\}$ such that $f(\ulcorner\varphi\urcorner) = 1$." Although this is Σ^1_1, every such f is provably unique, and thus $M \models \varphi$ is equivalent to a Π^1_1 formula. $\square$

Computable entailment thus transcends arithmetical truth, being recursively isomorphic to the Π^1_1 theory of the natural numbers, and also to membership in Kleene's $\mathcal{O}$, the set of notations for recursive ordinals. Nevertheless its complexity is towards the lower end of the logics considered by Väänänen [2001] and Koellner [2010], being for instance far less complex than the full second-order consequence relation. But as we shall soon see, such complexity is incompatible with the requirements of foundational analysis.

The Entscheidungsproblem was considered by Hilbert and others to be of such importance because a positive solution would have meant we could obtain, by finite means, knowledge of the provability or unprovability of all mathematical statements. The computational intractability of the classical provability relation constitutes an epistemic difficulty for mathematics. From this perspective, we should be troubled by an entailment relation such as Shore's with a far greater degree of uncomputability.

It's well known that truth definitions are not simple: Kripke's fixed-point construction of a truth predicate over the natural numbers is also Π^1_1-complete [Kripke 1975]. Provability, at least for classical first-order logic, is comparatively uncomplicated. If $\mathsf{RCA}_0 \vdash \varphi$ then we can produce a finitary proof witness by an exhaustive search. We have no such assurance when $\models_c \varphi$: computable entailment does not satisfy Gödel's completeness theorem, so we are unable to reduce this complex semantic relation to the more finitistically acceptable provability relation.

ω-logic does have a completeness theorem of sorts, namely the ω-completeness theorem of Henkin and Orey, which was stated towards the end of §5.1. By this theorem, restricting to ω-models is equivalent to closing one's consequence set under the ω-rule. This is typically formalised in terms of an infinitary proof calculus, where proofs are well-founded trees which branch infinitely on uses of the ω-rule. However, this completeness theorem does not induce a reduction in the complexity of the computable entailment relation: computable entailment is irredeemably infinitary.

Computable entailment is also, as should now be very clear, an impredicative relation. Shore's definition quantifies over all Turing ideals, and while theorem 5.3.2 shows that a definition quantifying only over countable Turing ideals is in fact equivalent to Shore's, computable entailment is still Π^1_1-complete, and thus an archetypal impredicative relation. As such the predicativist and the predicative reductionist (in the sense of §4.4) should not accept this relation as being well-defined, and should treat inferences that employ it with suspicion.

This brings us back to the theme of §5.2, namely the role of a reverse mathematical entailment relation within the foundational dialectic. If reverse mathematics is going to be a useful tool for foundational analysis of a given foundational theory—call it $\mathcal{F}$—then the entailment relation it employs had better be acceptable to $\mathcal{F}$-theorists, or they can simply reject any argument based on the foundational analysis thereby achieved as presupposing theoretical commitments which they reject.

So far we have discussed theoretical commitments in terms of the internal justificatory structure of foundational programmes, with a focus on the types of inferences allowed by the strictures of those programmes. In particular I have argued that a finitistic reductionist would not accept all the consequences which are computably entailed by a finitistically acceptable theory. But there are other dimensions of theoretical commitment, one of which concerns the proof-theoretic strength of the theory which an $\mathcal{F}$-theorist will accept and the commitments inherent in the use of computable entailment.

Consider the following principle which constitutes a nontriviality condition on computable entailment: Every set is contained in a countable Turing ideal. It is clear that if we want to use computational reverse mathematics then we should accept this principle, since we believe that while some statements in the language of second order arithmetic are computably entailed, not all are. This principle is not provable in RCA_0, but it is provable in ACA_0.[25]

Lemma 5.3.9. *The following is provable in* ACA_0. *Every set* $X \subseteq \mathbb{N}$ *is contained in a countable coded Turing ideal.*

Proof. Fix a universal lightface Π^0_1 formula $\pi(e, m, X)$ in the displayed free variables. Let $X \subseteq \mathbb{N}$, and let W be the set of triples $(m, (e_0, e_1))$ such that $\pi(e_1, m, X)$ and $\forall n(\neg\pi(e_0, n, X) \leftrightarrow \pi(e_1, n, X))$. W exists by arithmetical comprehension, and it is straightforward to check that it codes a countable Turing ideal M with $X \in M$. □

Given their acceptance of ACA_0, the predicativist will by lemma 5.3.9 accept the nontriviality of computable entailment. On the other hand, asserting the existence of a Π^1_1-complete relation such as computable entailment would exceed the existential boundaries which a predicativist should be comfortable with. Such a set can only be defined by a formula which universally quantifies over sets of natural numbers. It is thus thoroughly impredicative, as the predicativist can determine from within their own framework by showing that the existence of the truth set for X-computable entailment (for any $X \subseteq \mathbb{N}$) is equivalent to Π^1_1 comprehension.

Lemma 5.3.10. *The following is provable in* ACA_0. *Suppose M is a countable coded ω-model with satisfaction function f_M, and φ is an arithmetical formula with parameters from M. Then the following absoluteness fact holds:*

$$\varphi \leftrightarrow f(\ulcorner\varphi\urcorner) = 1$$

Proof. By induction on the complexity of φ. □

[25] It is actually provable in WKL_0, by a different proof technique to the one used here. My thanks to Carl Mummert for pointing this out to me.

The following theorem is then provable in the system known as ACA_0^+, which while still predicatively justifiable, extends ACA_0 with an axiom stating that the Turing jump can be iterated ω-many times.

Theorem 5.3.11. *The following are equivalent over* ACA_0^+*.*

1. Π^1_1 *comprehension.*
2. *For every* $X \subseteq \mathbb{N}$*, the truth set* $C(X)$ *of the* X*-computable entailment relation exists.*

Proof. As we saw in the proof of theorem 5.3.2, the truth set $C(X)$ for X-computable entailment has a Π^1_1 definition,

$$C(X) = \{ \ulcorner\varphi\urcorner \mid (\forall \text{ countable Turing ideals } M)(X \in M \to M \models \varphi) \}.$$

Let $X \subseteq \mathbb{N}$ be any set. Π^1_1-CA_0 proves $C(X)$ exists since it has a Π^1_1 definition in the parameter X. This completes the forward direction of the proof.

To prove the reversal we work in ACA_0^+. The following facts will be required.

(i) For any countable coded Turing ideal M and sentence φ of L_2, ACA_0^+ proves the existence and uniqueness of the valuation function $f : \mathrm{Sub}_M(\varphi) \to \{ 0, 1 \}$.

(ii) Arithmetical properties are absolute between any countable coded ω-model and the ambient model: that is, $\mathsf{ACA}_0 \vdash (\varphi(X) \leftrightarrow M \models \varphi(X))$ when φ is arithmetical.

Now, assuming that $C(X)$ exists for every $X \subseteq \mathbb{N}$, we prove the following principle known to be equivalent over RCA_0 to Π^1_1 comprehension: For any sequence of trees $\langle T_k \mid k \in \mathbb{N}\rangle$, $T_k \subseteq \mathbb{N}^{<\mathbb{N}}$, there exists a set Y such that $\forall k(k \in Y \leftrightarrow T_k$ has a path).

Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree. We prove the following claim: T is wellfounded if and only if every countable coded Turing ideal satisfies "T is wellfounded".

($\Rightarrow$) Suppose there is a countable coded Turing ideal M_1 which contains T and $M_1 \not\models \mathrm{WF}(T)$. M_1 thus contains a function $f : \mathbb{N} \to \mathbb{N}$ which is a path through T. T and f exist by recursive comprehension in the parameter M_1, so f witnesses the illfoundedness of T.

($\Leftarrow$) Suppose T is not wellfounded, so there is a path f through T. By lemma 5.3.9 there exists a countable coded Turing ideal M_2 that contains $T \oplus f$. Since M_2 contains f, and "f is a path through T" is arithmetical (and thus absolute), $M_2 \not\models \mathrm{WF}(T)$.

This completes the proof of the claim.

Let $Z = \langle T_k \mid k \in \mathbb{N} \rangle$ be a sequences of trees. By our initial assumption, $C(Z)$ exists, and thus the set $Y = \{\, k \mid \ulcorner \mathrm{WF}(T_k) \urcorner \notin C(Z) \,\}$ exists by recursive comprehension in the parameter $C(Z)$. By the claim it follows that for all k, $k \in Y \leftrightarrow T_k$ has a path. $\square$

Π^1_1-CA_0 is the strongest of the subsystems of second order arithmetic usually studied in reverse mathematics. Computational reverse mathematics therefore draws on resources which are unavailable in the four members of the Big Five that are proof-theoretically weaker than Π^1_1-CA_0.

Since theorem 5.3.11 is provable within a predicatively acceptable system, the predicativist is clearly in a position to calibrate the strength of the commitment involved in accepting computable entailment. Doing so, she will see that not only is it stronger than predicative systems like ACA_0, but also predicatively reducible ones like ATR_0. So not only does the existence of the truth set for the computable entailment relation exceed the strength of the predicativist and the predicative reductionist's theoretical resources, but they are in a position to see that it does. Since they reject impredicative mathematics, and thus reject Π^1_1 comprehension, they must therefore reject the equivalent statement that the truth set for computable entailment exists.

For foundational analysis to be a useful and worthwhile endeavour within the philosophy of mathematics, the fruits of its analysis must be epistemically available to disputants. Recall our example of Sarah the predicativist, whom we met in chapter 4. Since she accepts ACA_0, she believes that the equivalence between Π^1_1 comprehension and the statement "Every countable abelian group can be expressed as a direct sum of a divisible group and a reduced group" (hereafter referred to as P) is true, since it is provable in a system contained in ACA_0 (namely RCA_0). How she responds to Rebecca's challenge that Sarah's predicativism is misguided, since it does not allow her to prove this ordinary mathematical theorem, will depend on the details of her views about the foundations of mathematics, but crucially she cannot dismiss the equivalence as question-begging. On the other hand, suppose Rebecca were instead to present Sarah with the following argument: Π^1_1-CA_0 and P are computably equivalent, that is to say they are true in exactly the same Turing ideals. Sarah should therefore accept Π^1_1-CA_0, since P is an ordinary mathematical theorem that any decent foundational system should prove. In this case Sarah can resist the conclusion by refusing to accept the antecedent: computable equivalence is not a well-defined notion, since it presupposes theoretical resources which predicativism denies. Any argument presupposing that computable equivalence is a well-defined notion therefore begs the question against her position.

A philosophical argument that attempts to invoke reverse mathematical results in the context of foundational analysis should, if it is to have any force, appeal only to principles that the target of the argument already accepts. In other words, its presuppositions must not exceed their theoretical commitments. But the argument above shows that the theoretical commitments which accompany the use of computable entailment outstrip those acceptable to partisans of most of the foundational programmes analysable in reverse mathematics. Whatever its other virtues, computational reverse mathematics is an inappropriate setting in which to conduct foundational analysis.

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